

THREE-DIMENSIONAL HOMOGENEOUS SPACES WITH NON-SOLVABLE TRANSFORMATION GROUPS

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ABSTRACT. We classify all transitive actions of Lie algebras of vector fields on \mathbb{C}^3 and \mathbb{R}^3 up to a local equivalence and discuss why this classification can not be extended in general to the solvable case. The main technical tool is the structure of one-dimensional invariant foliations on homogeneous spaces.

1. INTRODUCTION

Classification of homogeneous spaces in low dimensions is a classical problem which goes back to Sophus Lie, who provided local classification of complex and real homogeneous spaces in dimensions 1 and 2. He also classified three classes of complex three-dimensional spaces, namely:

- primitive actions, i.e. actions without non-trivial invariant foliations;
- actions without invariant one-dimensional foliations;
- actions without invariant two-dimensional foliations.

For the rest of three-dimensional actions he provided a certain classification algorithm and claimed (as Fermat) that only lack of book space does not allow him to list the complete classification. Unfortunately, this classification was never published by him, but some parts of it appear in giant archive of handwritten papers, left after him [6]. As it is shown in this paper, it is unlikely that Sophus Lie really had explicitly written down this classification since for several cases it is hardly possible.

However, a large class of three-dimensional homogeneous spaces, namely, homogeneous spaces with non-solvable transformation group, allows explicit parametrization, and the main goal of this paper is to provide this classification. The main idea is to consider homogeneous spaces with one-dimensional invariant foliations as one-dimensional bundles over two-dimensional homogeneous spaces. As it was shown in [1] it can be assumed with minor exceptions that this bundle has a structure of vector bundle, and its transformation group is an extension of a transformation group of underlying two-dimensional space by means of a certain subspace in the space of all sections of this vector bundle.

Notice also, that the first attempt to classify all three-dimensional homogeneous spaces with non-solvable transformation groups was made by Morozov and his student in [2]. Unfortunately, his methods were rather complicated, and, as a result, a series of actions is missing in this work. Moreover,

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Morozov couldn't manage to provide necessary and sufficient conditions for different actions in his work to be equivalent. Thus, his classification lists contain equivalent actions, while in one case the essential parameter is missing.

The paper is organized as follows. In Section 2 we briefly recall the results of the work [1], where the structure of invariant one-dimensional foliations on homogeneous spaces is described. In Section 3 we summarize the results of Sophus Lie [7] on the classification of transitive Lie algebras of vector fields in the three-dimensional space that do not admit a one-dimensional invariant foliation. This includes all primitive actions, but also those that admit invariant 2-dimensional foliations with primitive actions on the fibers.

In Section 4 we carry out the full classification of transitive actions on \mathbb{C}^3 with non-solvable transformation groups that admit 1-dimensional foliation. We analyze the difference between real and complex case and provide the similar classification of actions on \mathbb{R}^3 on \mathbb{R}^3 in Section 5.

Finally, in Section 7 we provide one simple example of a class of transitive nilpotent transformation groups that do not admit an explicit parametrization by a finite set of parameters. In particular, this shows why similar classification of solvable Lie algebras of vector fields is a considerably more complex problem. The classification results are summarized in Appendix A.

2. ONE-DIMENSIONAL INVARIANT FOLIATIONS ON HOMOGENEOUS SPACES

2.1. Constructions of invariant one-dimensional foliations. In this subsection we describe local structure of one-dimensional invariant foliations on homogeneous spaces. Let $M = G/G_0$ be a homogeneous space of the Lie group G . Denote by o the point eG_0 , and by l_g the diffeomorphism of M defined by the action of the element $g \in G$. The *isotropy action* of G on $T_o(M)$ is a linear action defined by $g.v = d_o(l_g)(v)$ for $g \in G$, $v \in T_oM$. It is well-defined and supplies the tangent space T_oM with a G -module structure.

Let \mathfrak{g} be the Lie algebra of the Lie group G and \mathfrak{g}_0 the subalgebra of \mathfrak{g} corresponding to the subgroup G_0 . Locally at the point o , the homogeneous space M is completely determined by the pair $(\mathfrak{g}, \mathfrak{g}_0)$. We assume in the following that the action of G on M is *effective*, so that the identical element is the only element in G that acts trivially on M . This implies that the pair $(\mathfrak{g}, \mathfrak{g}_0)$ is also effective, i.e., \mathfrak{g}_0 does not contain any non-zero ideals of \mathfrak{g} . The tangent space T_oM can be identified with the quotient space $\mathfrak{g}/\mathfrak{g}_0$, and the isotropy action of G on T_oM with the adjoint action of G_0 on $\mathfrak{g}/\mathfrak{g}_0$.

It is well-known that invariant distributions on M are in one-to-one correspondence with submodules of the G -module $\mathfrak{g}/\mathfrak{g}_0$. Locally we may always assume that the subgroup G_0 is connected. Hence all submodules of the G_0 -module $\mathfrak{g}/\mathfrak{g}_0$ are in one-to-one correspondence with submodules of the \mathfrak{g}_0 -module $\mathfrak{g}/\mathfrak{g}_0$, where $x.(y + \mathfrak{g}_0) = [x, y] + \mathfrak{g}_0$, $x \in \mathfrak{g}_0$, $y \in \mathfrak{g}$.

Thus, the local description of one-dimensional invariant foliations on homogeneous spaces is equivalent to the description of triples $(\mathfrak{g}, \mathfrak{g}_0, W)$, where $(\mathfrak{g}, \mathfrak{g}_0)$ is an effective pair of Lie algebras and W is a one-dimensional submodule of the \mathfrak{g} -module $\mathfrak{g}/\mathfrak{g}_0$.

Let $N = H/H_0$ be an arbitrary n -dimensional homogeneous space. Let us introduce several ways of constructing a new $(n + 1)$ -dimensional homogeneous space supplied with a one-dimensional invariant foliation.

Example 1. Let G_0 be a Lie subgroup of H_0 of codimension 1. We set $G = H$ and $M = G/G_0$. Then $\pi: M \rightarrow N$, $hG_0 \mapsto hH_0$, is a fibre bundle with one-dimensional fibres. It is easy to see that these fibres form an invariant one-dimensional foliation on M .

Example 2. Let $\pi: M \rightarrow N$ be an H -invariant one-dimensional vector bundle, that is π is a vector bundle and there exists an action of H on M such that

- (1) $\pi(h.p) = h.\pi(p)$ for all $h \in H$, $p \in M$;
- (2) the mapping $h: \pi^{-1}(q) \rightarrow \pi^{-1}(h.q)$ is linear for all $h \in H$, $q \in N$.

Notice that H acts non-transitively on M , since, for example, the set of all zero vectors forms an orbit. But this action can be extended to a transitive one in the following way. Let $\mathcal{F}(\pi)$ be the space of all smooth sections of π . Consider the natural action of H on $\mathcal{F}(\pi)$:

$$(h.\alpha)(q) = h.\alpha(h^{-1}.q), \quad \alpha \in \mathcal{F}(\pi), \quad h \in H, \quad q \in N.$$

Let V be an arbitrary finite-dimensional submodule of the H -module $\mathcal{F}(\pi)$ and let $G = H \ltimes V$ be the semidirect product of H and V . We define the action of G on M in the following way:

$$(h, \alpha).p = h.p + \alpha(h.\pi(p)), \quad (h, \alpha) \in G, \quad p \in M.$$

It is easy to see that this action is well-defined and transitive on M , if V contains any non-zero section. The stationary subgroup G_0 of this action at the point $(o, 0)$ is equal to $H_0 \ltimes V_0$. This action is effective and preserves the fibres of the projection π , which form an invariant one-dimensional foliation on M .

Instead of semidirect product $H \ltimes V$ we can take an arbitrary extension G of H by means of the abelian subgroup V :

$$\{0\} \longrightarrow V \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow \{e\},$$

and an action of G on M such that

- (1) the following diagram is commutative:

$$\begin{array}{ccccccccc} \{0\} \times M & \longrightarrow & V \times M & \xrightarrow{\alpha \times \text{id}} & G \times M & \xrightarrow{\beta \times \pi} & H \times N & \longrightarrow & \{e\} \times N \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{\text{id}} & M & \xrightarrow{\text{id}} & M & \xrightarrow{\pi} & N & \xrightarrow{\text{id}} & N, \end{array}$$

(here all vertical arrows correspond to the actions of Lie groups on manifolds and V acts on M by parallel displacements on the fibers).

- (2) the mapping $g: \pi^{-1}(q) \rightarrow \pi^{-1}(\beta(g).q)$ is linear for all $g \in G$, $q \in N$.

Again, fibers of the projection π form an invariant one-dimensional foliation.

Example 3. As in the previous example, let $\pi: M \rightarrow N$ be an invariant one-dimensional vector bundle, and let V be a finite-dimensional submodule of the G -module $\mathcal{F}(\pi)$. We let $G = (H \times \mathbb{R}^*) \ltimes V$, where the elements of \mathbb{R}^*

act on V by scale transformations, and define the action of G on M in the following way:

$$(h, x, \alpha).p = h.p + x \alpha(h.\pi(p)), \quad (h, x, \alpha) \in G, \quad p \in M.$$

As in the previous example, the fibers of π form an one-dimensional invariant foliation.

Example 4. Let $G = H \times PSL(2, \mathbb{R})$, $M = N \times \mathbb{RP}^1$, and the action of G on M is a direct product of the action of H on N and the action of $PSL(2, \mathbb{R})$ on \mathbb{RP}^1 by projective transformations. This action is transitive and preserves fibres of the projection $\pi: M \rightarrow N$, $(q, p) \mapsto q$. The fibres of this projection form an invariant one-dimensional foliation.

Notice that if we take other one-dimensional homogeneous space instead of \mathbb{RP}^1 we get particular cases of the two previous examples.

The main result of this section is the following

Theorem 1. *Let $M = G/G_0$ be a homogeneous space with a fixed one-dimensional invariant foliation. Then there is a uniquely determined homogeneous space $N = H/H_0$ such that $\dim N = \dim M - 1$ and M is locally equivalent to one of the spaces constructed in Examples 1–4.*

The proof is based on several purely algebraic results presented in the next subsection.

2.2. Almost effective pairs of Lie algebras. Let us recall that an ideal of the Lie algebra \mathfrak{g} is called *characteristic*, if it is stable under all derivations of \mathfrak{g} . We call the pair $(\mathfrak{g}, \mathfrak{g}_0)$ *almost effective* if \mathfrak{g}_0 does not contain proper characteristic ideals of the Lie algebra \mathfrak{g} , and *maximal* if \mathfrak{g}_0 is a maximal subalgebra of \mathfrak{g} . Maximal pairs correspond to primitive homogeneous spaces, which do not preserve (even locally) any invariant foliations.

Lemma 1. *Any maximal almost effective pair $(\mathfrak{g}, \mathfrak{g}_0)$ over the field $k = \mathbb{R}$ or \mathbb{C} has one of the following forms:*

- (1) $\mathfrak{g} = \mathfrak{a} \ltimes (V \otimes k^n)$, $\mathfrak{g}_0 = \mathfrak{a}(V \otimes k^{n-1})$, where V is a faithful simple \mathfrak{a} -module and k^n is a trivial \mathfrak{a} -module;
- (2) \mathfrak{g} is a semisimple Lie algebra and \mathfrak{g}_0 is its maximal effective subalgebra.

Proof. The proof is based on the theory of Frattini subalgebras of Lie algebras. Let us recall that the *Frattini subalgebra* of a finite-dimensional Lie algebra \mathfrak{g} is defined to be the intersection of all maximal subalgebras of \mathfrak{g} and is denoted by $\phi(\mathfrak{g})$. It is clear that $\phi(\mathfrak{g})$ is stable under the group of all automorphisms of \mathfrak{g} and hence is a characteristic ideal of \mathfrak{g} .

Since \mathfrak{g}_0 is a maximal subalgebra of \mathfrak{g} , we have $\mathfrak{g}_0 \supset \phi(\mathfrak{g})$. Hence, \mathfrak{g}_0 can be almost effective only if $\phi(\mathfrak{g}) = 0$. Such Lie algebras were considered by E. Stitzinger [9]. He proved (Theorem 4 of [9]) that any finite-dimensional subalgebra \mathfrak{g} over a field of zero characteristic with $\phi(\mathfrak{g}) = \{0\}$ has the form $(\mathfrak{a} \ltimes W) \times \mathfrak{b}$, where W is an abelian ideal, \mathfrak{a} is a reductive subalgebra in $\mathfrak{gl}(W)$, and \mathfrak{b} is a semisimple ideal.

If $W = \{0\}$ then $\mathfrak{a} = \{0\}$ as well and \mathfrak{g} is semisimple. Since all ideals of semisimple Lie algebra are characteristic, we see that \mathfrak{g}_0 is maximal effective subalgebra of \mathfrak{g} .

Assume now that $W \neq \{0\}$. Since \mathfrak{g}_0 is maximal, we see that $\mathfrak{g}_0 + W = \mathfrak{g}$. Hence, $W_0 = W \cap \mathfrak{g}_0$ is an ideal of \mathfrak{g} and, in particular, is a submodule of the \mathfrak{a} -module W . Let $W = (V_1 \otimes k^{n_1}) + \cdots + (V_r \otimes k^{n_r})$ be the decomposition of W into the sum of isotypic components of the \mathfrak{a} -module W . Then each of this components is a characteristic ideal of \mathfrak{g} , and W_0 contains all these components apart from one of them. Hence, the \mathfrak{a} -module W has only one isotypic component: $W = V \otimes k^n$, and W_0 is equal to $V \otimes k^{n-1}$.

Consider now the projection $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/W = \mathfrak{a} \times \mathfrak{b}$. Since \mathfrak{g}_0 is maximal, we see that $\pi(\mathfrak{g}_0) = \mathfrak{a} \times \mathfrak{b}$ and the subalgebra \mathfrak{g}_0 has the form $\{x + \alpha(x) \mid x \in \mathfrak{a} \times \mathfrak{b}\} + W_0$, where $\alpha \in Z^1(\mathfrak{a} \times \mathfrak{b}, V)$. But it is easy to see that $H^1(\mathfrak{a} \times \mathfrak{b}, V) = \{0\}$ and we may assume $\alpha = 0$. Since \mathfrak{b} is a characteristic ideal of \mathfrak{g} , we see that $\mathfrak{b} = 0$ and the pair $(\mathfrak{g}, \mathfrak{g}_0)$ has the form (1) of the Lemma. \square

Corollary. *Any almost effective pair $(\mathfrak{g}, \mathfrak{g}_0)$ of codimension 1 has one the following forms:*

- (i) $\mathfrak{g} = k^n$, $\mathfrak{g}_0 = k^{n-1}$, $n \geq 1$;
- (ii) $\mathfrak{g} = (kE_n) \ltimes k^n$, $\mathfrak{g}_0 = (kE_n) \ltimes k^{n-1}$, $n \geq 1$;
- (iii) $\mathfrak{g} = \mathfrak{sl}(2, k)$, $\mathfrak{g}_0 = \mathfrak{sl}(2, k)$.

Proof. If the Lie algebra \mathfrak{g} is not semisimple then the pair $(\mathfrak{g}, \mathfrak{g}_0)$ has form (1) of the Lemma with $n \geq 0$. Since the codimension of \mathfrak{g}_0 is equal to the dimension of V , we see that $\dim V = 1$ and either $\mathfrak{a} = \{0\}$ or $\mathfrak{a} = k^*$. In the first case the pair $(\mathfrak{g}, \mathfrak{g}_0)$ has form (i), and in the second case it is if of the form (ii).

If \mathfrak{g} is semisimple, then \mathfrak{g}_0 is effective and the pair $(\mathfrak{g}, \mathfrak{g}_0)$ corresponds to some one-dimensional homogeneous space. But the only semisimple finite-dimensional subalgebra in the Lie algebra of vector fields on the line is $\mathfrak{sl}(2, k)$, and the pair $(\mathfrak{g}, \mathfrak{g}_0)$ in this case has form (iii). \square

Lemma 2. *Let $(\mathfrak{g}, \mathfrak{g}_0)$ be an effective pair and let \mathfrak{p} be a minimal overalgebra of the subalgebra \mathfrak{g}_0 . Suppose \mathfrak{a} the largest ideal of \mathfrak{g} lying in \mathfrak{p} , and $\mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{g}_0$. Then either $\mathfrak{a} = \{0\}$ and the pair $(\mathfrak{g}, \mathfrak{p})$ is also effective, or the pair $(\mathfrak{a}, \mathfrak{a}_0)$ is almost effective and has the same codimension as the pair $(\mathfrak{h}, \mathfrak{g}_0)$.*

Proof. Suppose that $\mathfrak{a} \neq \{0\}$. Let \mathfrak{b} be a proper characteristic ideal of \mathfrak{a} lying in \mathfrak{a}_0 . Then \mathfrak{b} is also an ideal of the Lie algebra \mathfrak{g} lying in \mathfrak{g}_0 , which contradicts the assumption that the pair $(\mathfrak{g}, \mathfrak{g}_0)$ is effective. This proves that the pair $(\mathfrak{a}, \mathfrak{a}_0)$ is almost effective.

Notice that $\mathfrak{g}_0 + \mathfrak{a} = \mathfrak{p}$, since \mathfrak{g}_0 does not contain \mathfrak{a} and $\mathfrak{g}_0 + \mathfrak{a}$ is an overalgebra of \mathfrak{g}_0 lying in \mathfrak{p} . Therefore the space $\mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{g}_0$ has the same codimension in \mathfrak{a} as \mathfrak{g}_0 is \mathfrak{p} . \square

Corollary. *Let \mathfrak{p} be an overalgebra of \mathfrak{g}_0 such that $\text{codim}_{\mathfrak{h}} \mathfrak{g}_0 = 1$. Then either it is effective subalgebra in \mathfrak{g} or the corresponding pair $(\mathfrak{a}, \mathfrak{a}_0)$ has one of the forms described in Corollary to Lemma 1*

2.3. Modules over pairs of Lie algebras. An algebraic version of invariant vector bundles over homogeneous spaces and modules of their sections is given by the notion of a *module over a pair of Lie algebras*.

Definition 1. Let $(\mathfrak{g}, \mathfrak{g}_0)$ be a pair of Lie algebras. A $(\mathfrak{g}, \mathfrak{g}_0)$ -*module* is a pair (V, V_0) where V is a \mathfrak{g} -module and V_0 is a subspace in V stable under

the action of \mathfrak{g}_0 . A module (V, V_0) is said to be *effective*, if V_0 does not contain any non-zero submodules of the \mathfrak{g} -module V .

By codimension of (V, V_0) we mean the codimension of V_0 in V , and we call the pair (V, V_0) finite-dimensional if so is V .

Let G/G_0 be effective homogeneous space and let $(\mathfrak{g}, \mathfrak{g}_0)$ be the corresponding pair of Lie algebras. Since we are interested in local equivalence of homogeneous spaces we may always suppose that G and G_0 are connected. It is well-known that all finite-dimensional invariant vector bundles $\pi: E \rightarrow M$ are described by finite-dimensional G_0 -modules. Indeed, to any such invariant vector bundle $\pi: E \rightarrow M$ we may correspond the G_0 -module E_o . Conversely, having such G_0 -module, we may reconstruct the whole bundle E by the formula $E = (G \times E_o)/G_0$ where G_0 acts on $G \times E_o$ as follows:

$$g_0 \cdot (g, v) = (gg_0^{-1}, g_0 \cdot v), \quad g_0 \in G_0, g \in G, v \in E_o.$$

The projection π maps the equivalence class of the point (g, v) to gG_0 in G/G_0 . Moreover, the space $\mathcal{F}(\pi)$ of all sections of π also has a nice interpretation in terms of G_0 -module E_o . Namely, $\mathcal{F}(\pi)$ can be identified with the induced G -module defined by

$$(1) \quad \text{Ind}_G(G_0, E_o) = \{\phi \in \text{Hom}(G, E_o) \mid (\phi(gg_0^{-1}) = g_0 \cdot \phi(g) \ \forall g_0 \in G_0)\}.$$

We call the subspace $V \subset \mathcal{F}(\pi)$ *non-degenerate*, if it acts transitively on fibers of π . Define the subspace $V_0 \subset V$ as

$$V_0 = \{\alpha \in V \mid \alpha(o) = 0\}.$$

Lemma 3. 1. The subspace V_0 is stable under the action of G_0 on V and contains no non-zero submodules of the G_0 -module V .

2. The G_0 -module V/V_0 is naturally isomorphic to the G_0 -module E_o .

Proof. The first two assertions are obvious. Suppose that W is a submodule of the H -module V lying in V_0 . Since H acts linearly on fibres of the projection π and transitively on the base N , we see that $W(q) = \{0\}$ for all points $q \in N$. Hence, $W = \{0\}$. \square

This lemma implies that (V, V_0) is an effective $(\mathfrak{g}, \mathfrak{g}_0)$ -module whose codimension coincides with the dimension of the vector bundle π . It is easy to see that the correspondence between the pairs (π, V) , where π is an invariant vector bundle over G/G_0 and V is a non-degenerate subspace in $\mathcal{F}(\pi)$, and effective $(\mathfrak{g}, \mathfrak{g}_0)$ -modules is one-to-one. Indeed, having a $(\mathfrak{g}, \mathfrak{g}_0)$ -module (V, V_0) we may construct the G_0 -module V/V_0 and the corresponding invariant vector bundle $\pi: E \rightarrow M$, such that $E_o = V/V_0$. Then equation (1) allows as to embed V to $\mathcal{F}(\pi)$ in the following way:

$$v \mapsto \phi_v, \text{ where } \phi_v: G \rightarrow E_o, \ g \mapsto g \cdot v + V_0.$$

It is easy to show that this mapping is injective if and only if (V, V_0) is effective.

2.4. Extensions of pairs of Lie algebras. Let (V, V_0) be a $(\mathfrak{g}, \mathfrak{g}_0)$ -module.

Definition 2. We say that a pair $(\mathfrak{h}, \mathfrak{h}_0)$ of Lie algebras is an *extension* of the $(\mathfrak{g}, \mathfrak{g}_0)$ -module (V, V_0) if

- (1) V is a commutative ideal in \mathfrak{h} , and $V_0 = V \cap \mathfrak{h}_0$;

(2) there exists a surjective homomorphism of Lie algebras $\alpha: \mathfrak{h} \rightarrow \mathfrak{g}$ such that

- i) $\ker \alpha = V$;
- ii) $\alpha(\mathfrak{h}_0) = \mathfrak{g}_0$;
- iii) $[x, v] = \alpha(x) \cdot v$ for all $x \in \mathfrak{h}$, $v \in V$.

Let $\alpha: \mathfrak{h} \rightarrow \mathfrak{g}$ be a homomorphism satisfying condition (2) of Definition 2, and let $\alpha_0 = \alpha|_{\mathfrak{h}_0}$. Then conditions (2.i) and (2.ii) of this definition actually mean that we have the following commutative diagram:

$$\begin{array}{ccccccccc} \{0\} & \longrightarrow & V_0 & \longrightarrow & \mathfrak{h}_0 & \xrightarrow{\alpha_0} & \mathfrak{g}_0 & \longrightarrow & \{0\} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \{0\} & \longrightarrow & V & \longrightarrow & \mathfrak{h} & \xrightarrow{\alpha} & \mathfrak{g} & \longrightarrow & \{0\}, \end{array}$$

where all vertical arrows are natural embeddings. Equivalence of two extensions can be defined in the natural way.

Consider the following subcomplex of the standard cohomology complex $C(\mathfrak{g}, V)$:

$$C^n(\mathfrak{g}, \mathfrak{g}_0, V, V_0) = \{\omega \in C^n(\mathfrak{g}, V) \mid \omega(\wedge^n \mathfrak{g}_0) \subset V_0\}.$$

It is easy to check that $dC^n(\mathfrak{g}, \mathfrak{g}_0, V, V_0) \subset C^{n+1}(\mathfrak{g}, \mathfrak{g}_0, V, V_0)$, and therefore, our subcomplex is well-defined. We call the cohomology space $H(\mathfrak{g}, \mathfrak{g}_0, V, V_0)$ of this subcomplex *the cohomology space of the pair $(\mathfrak{g}, \mathfrak{g}_0)$ with values in (V, V_0)* .

Theorem 2.

1. *Equivalence classes of extensions of the $(\mathfrak{g}, \mathfrak{g}_0)$ -module (V, V_0) are in one-to-one correspondence with elements of the second cohomology space $H^2(\mathfrak{g}, \mathfrak{g}_0, V, V_0)$.*

2. *Let $\pi: C(\mathfrak{g}, V) \rightarrow C(\mathfrak{g}_0, V/V_0)$ be the natural projection of complexes given by*

$$\pi(\omega)(x_1, \dots, x_n) = \omega(x_1, \dots, x_n) + V_0, \quad \omega \in C^n(\mathfrak{g}, V), \quad x_1, \dots, x_n \in \mathfrak{g}_0.$$

Then the following sequence of complexes is exact:

$$(2) \quad \{0\} \rightarrow C(\mathfrak{g}, \mathfrak{g}_0, V, V_0) \rightarrow C(\mathfrak{g}, V) \rightarrow C(\mathfrak{g}_0, V/V_0) \rightarrow \{0\}.$$

Proof.

1. Let β be a section of the surjection α such that $\beta(\mathfrak{g}_0) \subset \mathfrak{h}_0$. Consider the element $\omega \in C(\mathfrak{g}, \mathfrak{g}_0, V, V_0)$ such that:

$$\omega(x, y) = \beta([x, y]) - [\beta(x), \beta(y)], \quad x, y \in \mathfrak{g}.$$

We see that $\alpha(\omega(x, y)) = 0$, and if $x, y \in \mathfrak{g}_0$, then $\omega(x, y) \in V_0$. Therefore, ω is well-defined. A standard computation shows that $d\omega = 0$, and hence $\omega \in Z^2(\mathfrak{g}, \mathfrak{g}_0, V, V_0)$. Any other section β' of α with $\beta'(\mathfrak{g}_0) \subset \mathfrak{h}_0$ has the form $\beta' = \beta + \phi$, where $\phi \in C^1(\mathfrak{g}, \mathfrak{g}_0, V, V_0)$. The corresponding element ω' is equal to $\omega + d\phi$. This proves the first part of the Theorem.

2. This is an immediate consequence of the definitions. \square

The short exact sequence (2) of complexes produces the long exact sequence of their cohomology spaces. As an immediate application of this, we obtain the following

Corollary. *Suppose that the Lie algebra \mathfrak{g} is semisimple. Then equivalence classes of extensions of the $(\mathfrak{g}, \mathfrak{g}_0)$ -module (V, V_0) are in one-to-one correspondence with elements of the space $H^1(\mathfrak{g}_0, V/V_0)$.*

Proof. Since \mathfrak{g} is semisimple, we have $H^1(\mathfrak{g}, V) = H^2(\mathfrak{g}, V) = \{0\}$. But from the long exact sequence

$$\cdots \rightarrow H^1(\mathfrak{g}, V) \rightarrow H^1(\mathfrak{g}_0, V/V_0) \rightarrow H^2(\mathfrak{g}, \mathfrak{g}_0, V, V_0) \rightarrow H^2(\mathfrak{g}, V) \rightarrow \cdots$$

we see that the spaces $H^2(\mathfrak{g}, \mathfrak{g}_0, V, V_0)$ and $H^1(\mathfrak{g}_0, V/V_0)$ are naturally isomorphic. \square

2.5. Proof of Theorem 1. Local description of one-dimensional invariant distributions on homogeneous spaces is equivalent to the description of triples $(\mathfrak{g}, \mathfrak{g}_0, W)$, where $(\mathfrak{g}, \mathfrak{g}_0)$ is the effective pair corresponding to a given homogeneous space, and W is the one-dimensional submodule of the \mathfrak{g}_0 -module $\mathfrak{g}/\mathfrak{g}_0$ corresponding to the invariant distribution.

Let $\pi: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{g}_0$ be the natural projection, and $\mathfrak{p} = \pi^{-1}(W)$. Then \mathfrak{p} is a minimal overalgebra of \mathfrak{g}_0 . Let \mathfrak{a} be the largest ideal of the Lie algebra \mathfrak{g} lying in \mathfrak{p} . We let $\mathfrak{a}_0 = \mathfrak{a} \cap \mathfrak{g}_0$. From the Corollary to Lemma 2 it follows that we have one and only one of the following cases:

- I. The pair $(\mathfrak{g}, \mathfrak{p})$ is effective.
- II. $\mathfrak{a} = k^n$, $\mathfrak{a}_0 = k^{n-1}$, $(n \geq 1)$.
- III. $\mathfrak{a} = (kE_n) \ltimes k^n$, $\mathfrak{a}_0 = (kE_n) \ltimes k^{n-1}$, $(n \geq 1)$.
- IV. $\mathfrak{a} = \mathfrak{sl}(2, k)$, $\mathfrak{a}_0 = \mathfrak{st}(2, k)$.

Consider separately each of these cases. In case I the local structure of the corresponding invariant one-dimensional distribution is described in Example 1.

Lemma 4. *In case II all triples $(\mathfrak{g}, \mathfrak{g}_0, W)$ can be described as follows. There exist an effective pair $(\mathfrak{h}, \mathfrak{h}_0)$ of codimension $\text{codim}_{\mathfrak{g}} \mathfrak{g}_0 - 1$ and an effective $(\mathfrak{h}, \mathfrak{h}_0)$ -module (V, V_0) of codimension 1 such that*

- (1) *the pair $(\mathfrak{g}, \mathfrak{g}_0)$ is an extension of the $(\mathfrak{h}, \mathfrak{h}_0)$ -module (V, V_0) ;*
- (2) *$W = (V + \mathfrak{g}_0)/\mathfrak{g}_0$.*

Proof. Put $V = \mathfrak{a}$, $V_0 = \mathfrak{a}_0$, $\mathfrak{h} = \mathfrak{g}/\mathfrak{a}$, and $\mathfrak{h}_0 = \mathfrak{p}/\mathfrak{a}_0$. Then the pair $(\mathfrak{h}, \mathfrak{h}_0)$ is effective and its codimension is one less than the codimension of $(\mathfrak{g}, \mathfrak{g}_0)$. Since the ideal \mathfrak{a} is commutative, the space V can be naturally supplied with the structure of an \mathfrak{h} -module. The condition $\mathfrak{p} = \mathfrak{a} + \mathfrak{g}_0$ implies that the subspace $V_0 = \mathfrak{a} \cap \mathfrak{g}_0$ is invariant under the action of \mathfrak{h}_0 on V , which means that the pair (V, V_0) is an $(\mathfrak{h}, \mathfrak{h}_0)$ -module. Since the pair $(\mathfrak{g}, \mathfrak{g}_0)$ is effective, we conclude that the $(\mathfrak{h}, \mathfrak{h}_0)$ -module (V, V_0) is also effective. Finally, since $W = \mathfrak{p}/\mathfrak{g}_0$ and $\mathfrak{p} = \mathfrak{a} + \mathfrak{g}_0$, we have $W = (V + \mathfrak{g}_0)/\mathfrak{g}_0$. \square

The local structure of the corresponding invariant one-dimensional distribution is described in Example 2.

Lemma 5. *In case III all triples $(\mathfrak{g}, \mathfrak{g}_0, W)$ can be described as follows. There exist an effective pair $(\mathfrak{h}, \mathfrak{h}_0)$ of codimension $\text{codim}_{\mathfrak{g}} \mathfrak{g}_0 - 1$ and an effective $(\mathfrak{h} \times k, \mathfrak{h}_0 \times k)$ -module (V, V_0) of codimension 1 such that*

- (1) *$(\mathfrak{g}, \mathfrak{g}_0)$ is the trivial extension of the $(\mathfrak{h} \times k, \mathfrak{h}_0 \times k)$ -module (V, V_0) (i.e., $\mathfrak{g} = (\mathfrak{h} \times k) \ltimes V$, $\mathfrak{g}_0 = (\mathfrak{h}_0 \times k) \ltimes V_0$);*

- (2) $(0, x).v = xv$ for all $x \in k$, $v \in V$;
- (3) $W = (V + \mathfrak{g}_0)/\mathfrak{g}_0$.

Proof. Let $V = [\mathfrak{a}, \mathfrak{a}]$ and $V_0 = V \cap \mathfrak{g}_0$. Then V is a characteristic commutative ideal of the Lie algebra \mathfrak{a} and, therefore, is an ideal of \mathfrak{g} . Since V is commutative, it can be naturally supplied with the structure of a \mathfrak{g}/V -module. Let $u = E_n + V \in \mathfrak{g}/V$. Then ku is an ideal in \mathfrak{g}/V . Consider the following homomorphism of Lie algebras:

$$\phi: \mathfrak{g}/V \rightarrow k, \quad x \mapsto \text{tr}(x_V).$$

Let $\mathfrak{h} = \ker \phi$ and $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{p}/V$. Since $u.v = v$ for all $v \in V$, we see that $u \notin \ker \phi$ and that the Lie algebra \mathfrak{g}/V is the direct sum of the ideals \mathfrak{h} and ku . Since $u \in \mathfrak{p}/V$, we have $\mathfrak{p}/V + \mathfrak{h} = \mathfrak{g}/V$. It follows that $\mathfrak{p}/V = \mathfrak{h}_0 \oplus ku$, and the codimension of the pair $(\mathfrak{h}, \mathfrak{h}_0)$ is one less than that of the pair $(\mathfrak{g}, \mathfrak{g}_0)$.

Then, as in the proof of Lemma 4, we obtain that the pair $(\mathfrak{h}, \mathfrak{h}_0)$ is effective, the $(\mathfrak{h} \times k, \mathfrak{h}_0 \times k)$ -module (V, V_0) is also effective and has codimension 1, and $W = (V + \mathfrak{g}_0)/\mathfrak{g}_0$.

Let us prove that any extension of the $(\mathfrak{h} \times k, \mathfrak{h}_0 \times k)$ -module (V, V_0) is trivial. Let as before $u = (0, 1)$. Then it determines an internal gradations on $\mathfrak{h} \times k$ and V such that all elements in $\mathfrak{h} \times k$ have degree 0 and all elements of V have degree 1. It follows from [3, Theorem 1.5.2a] that $H(\mathfrak{h} \times k, V) = H(\mathfrak{h}_0 \times k, V/V_0) = \{0\}$. Now Theorem 2 implies that second cohomology space $H^2(\mathfrak{h} \times k, \mathfrak{h}_0 \times k, V, V_0)$ is also trivial. \square

The local structure of the corresponding invariant one-dimensional distribution is described in Example 3.

Lemma 6. *In case IV there exists an effective pair $(\mathfrak{h}, \mathfrak{h}_0)$ of codimension $\text{codim}_{\mathfrak{g}} \mathfrak{g}_0 - 1$ such that the triple $(\mathfrak{g}, \mathfrak{g}_0, W)$ has the following form:*

- (1) $\mathfrak{g} = \mathfrak{h} \times \mathfrak{sl}(2, k)$;
- (2) $\mathfrak{g}_0 = \mathfrak{h}_0 \times \mathfrak{st}(2, k)$;
- (3) $W = (\mathfrak{h}_0 \times \mathfrak{sl}(2, k))/\mathfrak{g}_0$.

Proof. Since the ideal $\mathfrak{a} = \mathfrak{sl}(2, k)$ is simple, there exists an ideal \mathfrak{h} in \mathfrak{g} complementary to \mathfrak{a} . Let $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{p}$. Since $\mathfrak{a} \subset \mathfrak{p}$, we see that the pair $(\mathfrak{h}, \mathfrak{h}_0)$ is effective and has codimension $\text{codim}_{\mathfrak{g}} \mathfrak{g}_0 - 1$. Let us show that $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{st}(2, k)$. The projection of \mathfrak{g}_0 on \mathfrak{a} cannot be larger than $\mathfrak{a}_0 = \mathfrak{st}(2, k)$. Indeed, if $x \in \mathfrak{h}_0$, $y \in \mathfrak{a}$, and $x+y \in \mathfrak{g}_0$, then $y \in [y+\mathfrak{a}_0, \mathfrak{a}_0] = [x+y+\mathfrak{a}_0, \mathfrak{a}_0] \subset \mathfrak{g}_0$. But since \mathfrak{g}_0 has codimension 1 in \mathfrak{p} , we have $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{st}(2, k)$. The rest of the proof is trivial. \square

The local structure of the corresponding invariant one-dimensional distribution is described in Example 4.

This completes the proof of Theorem 1. \square

3. LIE'S RESULTS ON THE CLASSIFICATION OF LIE ALGEBRAS OF VECTOR FIELDS

The local classification of all Lie algebras on the plane, both over the fields \mathbb{R} and \mathbb{C} , was obtained by Sophus Lie [7]. We list only transitive Lie algebras in Subsection A.2.

Let (G, M) be a three-dimensional homogeneous space, and let the Lie group G be connected and non-solvable. We fix an arbitrary point $o \in M$ and denote by G_0 the stationary subgroup at the point o . Since we are interested only in the local equivalence problem, we can assume without loss of generality that both G and G_0 are connected.

Consider the corresponding pair $(\mathfrak{g}, \mathfrak{g}_0)$ of Lie algebras and the isotropy \mathfrak{g}_0 -module $\mathfrak{g}/\mathfrak{g}_0$. There are only two cases possible:

- (1) The \mathfrak{g}_0 -module $\mathfrak{g}/\mathfrak{g}_0$ does not have any one-dimensional submodules;
- (2) There exists a one-dimensional submodule W of the \mathfrak{g}_0 -module $\mathfrak{g}/\mathfrak{g}_0$.

The first case was considered by Sophus Lie (even more widely in the non-homogeneous situation). His result can now be formulated as follows:

Theorem 3. *All effective pairs $(\mathfrak{g}, \mathfrak{g}_0)$ of codimension three, such that the isotropy module does not have any one-dimensional submodules, are equivalent to one and only one of the following:*

primitive pairs:

- (1) $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$, $\mathfrak{g}_0 = \left\{ \begin{pmatrix} X & Y \\ 0 & -\text{tr } X \end{pmatrix} \middle| \begin{array}{l} X \in \mathfrak{gl}(3, \mathbb{C}), \\ Y \in \mathbb{C}^3 \end{array} \right\};$
- (2) $\mathfrak{g} = \mathfrak{gl}(3, \mathbb{C}) \ltimes \mathbb{C}^3$, $\mathfrak{g}_0 = \mathfrak{gl}(3, \mathbb{C});$
- (3) $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C}) \ltimes \mathbb{C}^4$, $\mathfrak{g}_0 = \mathfrak{sl}(3, \mathbb{C});$
- (4) $\mathfrak{g} = \mathfrak{so}(5, \mathbb{C}) = \left\{ \begin{pmatrix} y & {}^tT & 0 \\ Z & X & T \\ 0 & {}^tZ & -y \end{pmatrix} \middle| \begin{array}{l} X + {}^tX = 0, \ y \in \mathbb{C}, \\ Z, T \in \mathbb{C}^3 \end{array} \right\},$
 $\mathfrak{g}_0 = \left\{ \begin{pmatrix} y & {}^tT & 0 \\ 0 & X & T \\ 0 & 0 & -y \end{pmatrix} \right\};$
- (5) $\mathfrak{g} = \mathfrak{co}(3, \mathbb{C}) \ltimes \mathbb{C}^3$, $\mathfrak{g}_0 = \mathfrak{co}(3, \mathbb{C});$
- (6) $\mathfrak{g} = \mathfrak{so}(4, \mathbb{C})$, $\mathfrak{g}_0 = \mathfrak{so}(3, \mathbb{C}) = \left\{ \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \middle| X - {}^tX = 0 \right\};$
- (7) $\mathfrak{g} = \mathfrak{so}(3, \mathbb{C}) \ltimes \mathbb{C}^3$, $\mathfrak{g}_0 = \mathfrak{so}(3, \mathbb{C});$
- (8) $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{C}) = \left\{ \begin{pmatrix} x_1 & x_2 & z_1 & z_2 \\ x_3 & x_4 & z_3 & z_1 \\ y_1 & y_2 & -x_4 & -x_2 \\ y_3 & y_1 & -x_3 & -x_1 \end{pmatrix} \middle| x_i, y_i, z_i \in \mathbb{C} \right\},$
 $\mathfrak{g}_0 = \left\{ \begin{pmatrix} x_1 & x_2 & z_1 & z_2 \\ 0 & x_4 & z_3 & z_1 \\ 0 & y_2 & -x_4 & -x_2 \\ 0 & 0 & 0 & -x_1 \end{pmatrix} \middle| x_1, x_2, x_4, y_2, z_1, z_2, z_3 \in \mathbb{C} \right\},$

imprimitive pairs:

- (9) $\mathfrak{g} = (\mathfrak{sl}(2, \mathbb{C}) \times \langle \frac{\partial}{\partial x} \rangle) \ltimes (\mathbb{C}^2 \otimes V(p(t))),$
 $\mathfrak{g}_0 = (\mathfrak{sl}(2, \mathbb{C}) \times \{0\}) \ltimes (\mathbb{C}^2 \otimes V_0(p(t)));$

where $p(t) = a_0 + a_1t + \dots + a_nt^n$ is an arbitrary polynomial in t , $V(p(t))$ is the space of solutions of the corresponding linear differential equation

$$a_nf^{(n)}(x) + \dots + a_1f'(x) + a_0f(x) = 0,$$

and $V_0(p(t))$ consists of all functions $f \in V(p(t))$ such that $f(0) = 0$.

$$(10) \quad \mathfrak{g} = (\mathfrak{gl}(2, \mathbb{C}) \times \langle \frac{\partial}{\partial x} \rangle) \ltimes (\mathbb{C}^2 \otimes V(p(t))),$$

$$\mathfrak{g}_0 = (\mathfrak{gl}(2, \mathbb{C}) \times \{0\}) \ltimes (\mathbb{C}^2 \otimes V_0(p(t)));$$

$$(11) \quad \mathfrak{g} = (\mathfrak{sl}(2, \mathbb{C}) \times \langle \frac{\partial}{\partial x}, x \frac{\partial}{\partial x} + \alpha \rangle) \ltimes (\mathbb{C}^2 \otimes V(t^n)),$$

$$\mathfrak{g}_0 = (\mathfrak{sl}(2, \mathbb{C}) \times \langle x \frac{\partial}{\partial x} + \alpha \rangle) \ltimes (\mathbb{C}^2 \otimes V_0(t^n));$$

$$(12) \quad \mathfrak{g} = (\mathfrak{gl}(2, \mathbb{C}) \times \langle \frac{\partial}{\partial x}, x \frac{\partial}{\partial x} \rangle) \ltimes (\mathbb{C}^2 \otimes V(t^n)),$$

$$\mathfrak{g}_0 = (\mathfrak{gl}(2, \mathbb{C}) \times \langle x \frac{\partial}{\partial x} \rangle) \ltimes (\mathbb{C}^2 \otimes V_0(t^n));$$

$$(13) \quad \mathfrak{g} = (\mathfrak{sl}(2, \mathbb{C}) \times \langle \frac{\partial}{\partial x}, 2x \frac{\partial}{\partial x} + n, x^2 \frac{\partial}{\partial x} + nx \rangle) \ltimes (\mathbb{C}^2 \otimes V(t^{n+1})),$$

$$\mathfrak{g}_0 = (\mathfrak{sl}(2, \mathbb{C}) \times \langle 2x \frac{\partial}{\partial x} + n, x^2 \frac{\partial}{\partial x} + nx \rangle) \ltimes (\mathbb{C}^2 \otimes V_0(t^{n+1}));$$

$$(14) \quad \mathfrak{g} = (\mathfrak{gl}(2, \mathbb{C}) \times \langle \frac{\partial}{\partial x}, x \frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial x} \rangle) \ltimes (\mathbb{C}^2 \otimes V(t^{n+1})),$$

$$\mathfrak{g}_0 = (\mathfrak{gl}(2, \mathbb{C}) \times \langle x \frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial x} \rangle) \ltimes (\mathbb{C}^2 \otimes V_0(t^{n+1})).$$

We list the corresponding Lie algebras of vector fields in complex space in Subsections A.3 and A.4.

4. NON-SOLVABLE LIE ALGEBRAS IN SPACE WITH AN INVARIANT ONE-DIMENSIONAL FOLIATION

4.1. Possible cases. Consider now the case, when the isotropy module has some one-dimensional submodule. Let \mathfrak{p} be an overalgebra of \mathfrak{g}_0 , such that $\text{codim}_{\mathfrak{h}} \mathfrak{g}_0 = 1$, and let \mathfrak{a} be a maximal ideal of \mathfrak{g} contained in \mathfrak{p} . From Theorem 1 it follows that only 4 cases are possible:

[B] $\mathfrak{a} = \{0\}$; description of such pairs $(\mathfrak{g}, \mathfrak{g}_0)$ is reduced to the classification of subalgebras of codimension 1 in stationary subalgebras of homogeneous spaces [6–14].

[C1] \mathfrak{a} is commutative; then the pair $(\mathfrak{g}, \mathfrak{g}_0)$ is an extension of one of the pairs [6–14] by means of a certain effective $(\mathfrak{g}, \mathfrak{g}_0)$ -module (V, V_0) of codimension 1.

[C2] \mathfrak{a} is isomorphic to $(k \text{ id}) \ltimes k^n$; this case is similar to the previous one with one additional simplification. Namely, all extensions are trivial in this case.

[D] \mathfrak{a} is isomorphic to $\mathfrak{sl}(2, k)$. In this case the pair $(\mathfrak{g}, \mathfrak{g}_0)$ has the form $\mathfrak{g} = \mathfrak{h} \times \mathfrak{sl}(2, k)$, $\mathfrak{g}_0 = \mathfrak{h}_0 \times \mathfrak{sl}(2, k)$, where $(\mathfrak{h}, \mathfrak{h}_0)$ is an effective pair of Lie algebras corresponding to a certain two-dimensional homogeneous space. Since the description of such pairs is known, this case does not require any additional work.

4.2. Subalgebras of codimension 1 in stationary subalgebras. Let \mathfrak{g} be one of the Lie algebras of vector fields [6–14] listed in A.2, and let \mathfrak{h}_0 be its stationary subalgebra. In this subsection we list all subalgebras \mathfrak{g}_0 in \mathfrak{h}_0 of codimension 1, which satisfy one additional property: *there are no ideals \mathfrak{a} in \mathfrak{g} such that $\text{codim}_{\mathfrak{a}+\mathfrak{g}_0} \mathfrak{g}_0 = 1$* . This property guarantees that the pair $(\mathfrak{g}, \mathfrak{g}_0)$ will not fall to three other cases C1, C2, D described above.

All pairs $(\mathfrak{g}, \mathfrak{g}_0)$ which satisfy this property have the form:

$$6, 7: \quad \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}), \quad \mathfrak{g}_0 = \{0\};$$

$$8: \quad \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C}), \quad \mathfrak{g}_0 = \left\{ \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} + \begin{pmatrix} \alpha x & z \\ 0 & -\alpha x \end{pmatrix} \right\};$$

$$9a: \quad \mathfrak{g} = \mathfrak{sl}(3, \mathbb{C}), \quad \mathfrak{g}_0 = \left\{ \begin{pmatrix} x & y & u \\ z & -x & v \\ 0 & 0 & 0 \end{pmatrix} \right\};$$

$$9b: \mathfrak{g} = \mathfrak{sl}(3, \mathbb{C}), \mathfrak{g}_0 = \left\{ \begin{pmatrix} x & z & u \\ 0 & y & v \\ 0 & 0 & -x-y \end{pmatrix} \right\};$$

$$10: \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \ltimes \mathbb{C}^2, \mathfrak{g}_0 = \left\{ \begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} \right\} \times \{0\};$$

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - 2z \frac{\partial}{\partial z}, y \frac{\partial}{\partial x} - z^2 \frac{\partial}{\partial z} \right\rangle.$$

This case is equivalent to the case 17a for $n = 1$;

$$11a: \mathfrak{g} = \mathfrak{gl}(2, \mathbb{C}) \ltimes \mathbb{C}^2, \mathfrak{g}_0 = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right\} \times \{0\};$$

$$11b: \mathfrak{g} = \mathfrak{gl}(2, \mathbb{C}) \ltimes \mathbb{C}^2, \mathfrak{g}_0 = \left\{ \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \right\} \times \{0\};$$

$$13a: \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \ltimes V_n = \langle x, y, h, e_0, \dots, e_n \rangle, \mathfrak{g}_0 = \langle x, h, e_0, \dots, e_{n-2} \rangle;$$

$$13b: \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C}) \ltimes V_2, \mathfrak{g}_0 = \langle x, h + e_1, e_0 \rangle;$$

$$14: \mathfrak{g} = \mathfrak{gl}(2, \mathbb{C}) \ltimes V_n, \mathfrak{g}_0 = \mathfrak{t}(2, \mathbb{C}) \ltimes \langle e_0, \dots, e_{n-2} \rangle.$$

The corresponding Lie algebras of vector fields are listed in Subsection A.5.

4.3. Extensions of pairs of codimension 2. Now consider the cases C1 and C2. The solution of the problem in these cases can be divided to the following steps:

- I. Description of all invariant one-dimensional vector bundles over the homogeneous spaces [6-14].
- II. Description of all finite-dimensional submodules in the corresponding infinite-dimensional modules of all sections of all vector bundles from I.
- III. Computation of cohomology spaces which describe all possible extensions of pairs [6-14] by means of submodules from II (only for C1).

6. I. $\langle \frac{\partial}{\partial x}, 2x \frac{\partial}{\partial x} - \frac{\partial}{\partial y}, x^2 \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + \alpha e^{-2y} \rangle.$

II. Finite-dimensional invariant subspaces exist only for $\alpha = 0$ and have the form:

$$V_m = \langle x^i e^{my} \mid 0 \leq i \leq m \rangle, \quad m \in \mathbb{N} \cup \{0\}.$$

III. Since \mathfrak{g} is semisimple, the cohomology space $H^2((\mathfrak{g}, \mathfrak{g}_0), (V, V_0))$ is isomorphic to $H^1(\mathfrak{g}_0, V/V_0)$. Hence, this space is non-trivial, if and only if the action of \mathfrak{g}_0 to V/V_0 is trivial. In our case

$$V = V_{m_1} \oplus \dots \oplus V_{m_k}, \quad 0 \leq m_1 < \dots < m_k,$$

$$V_0 = \{f \in V \mid f(0, 0) = 0\}.$$

It is easy to check that in all these cases $\mathfrak{g}_0.V \subset V_0$. Hence, the required cohomology space is always one-dimensional, and the corresponding twisted Lie algebra of vector fields is given in Subsection A.6 under number C1c.

7. I. $\langle \frac{\partial}{\partial x}, x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} - \alpha/2, -x^2 \frac{\partial}{\partial x} + (1 + 2xy) \frac{\partial}{\partial y} + \alpha x \rangle.$

II. Finite-dimensional subspaces exist only if $\alpha = n$, where $n \in \mathbb{N} \cup \{0\}$, and can be described in the following way:

$$V_{n,m} = \langle \frac{d^i}{dx^i} (x^{n+m} (1 + xy)^m) \mid 0 \leq i \leq n + 2m \rangle, \quad m \in \mathbb{N} \cup \{0\}.$$

III. As in the previous case, the cohomology space $H^2((\mathfrak{g}, \mathfrak{g}_0), (V, V_0))$ is non-trivial if and only if the action of \mathfrak{g}_0 on V/V_0 is trivial. This happens only for $n = 0$, and in this case the cohomology space is one-dimensional. The corresponding twisted pairs are C2c.

8. I. $\langle \frac{\partial}{\partial x}, 2x\frac{\partial}{\partial x} - \alpha, x^2\frac{\partial}{\partial x} - \alpha x, \frac{\partial}{\partial y}, 2y\frac{\partial}{\partial y} - \beta, y^2\frac{\partial}{\partial y} - \beta y \rangle$.

II. Finite-dimensional subspaces exist only if $\alpha, \beta \in \mathbb{N} \cup \{0\}$. In this case it is unique and has the form:

$$V = \langle x^i y^j \mid 0 \leq i \leq \alpha, 0 \leq j \leq \beta \rangle.$$

The cohomology spaces are non-trivial only in the case when $\alpha = 0$ or $\beta = 0$. In cases $\alpha = 0, \beta \neq 0$ and $\alpha \neq 0, \beta = 0$ the cohomology space is one-dimensional, and the corresponding twisted pair of Lie algebras has the form:

$$\langle \frac{\partial}{\partial x}, x\frac{\partial}{\partial x}, x^2\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial y}, y^2\frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle.$$

If $\alpha = \beta = 0$, then the cohomology space is two-dimensional, and we get one more Lie algebra of vector fields C3c.

9. I. $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{2\alpha}{3}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y}, x\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \alpha\right), y\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \alpha\right) \rangle$.

II. Finite-dimensional submodules exist only if $\alpha \in \mathbb{N} \cup \{0\}$. In this case it is unique and has the form:

$$V = \langle x^i y^j \mid 0 \leq i + j \leq \alpha \rangle.$$

III. Since the Lie algebra \mathfrak{g} is semisimple, the required cohomology space is isomorphic to $H^1(\mathfrak{g}_0, V/V_0)$. This space is trivial for $\alpha > 0$ and one-dimensional for $\alpha = 0$. The corresponding Lie algebra of vector fields is C4c.

10. I. $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y} \rangle$.

II. All finite-dimensional submodules have the form:

$$V_m = \langle x^i y^j \mid 0 \leq i + j \leq m \rangle, \quad m \in \mathbb{N} \cup \{0\}.$$

III. The cohomology space $H^2((\mathfrak{g}, \mathfrak{g}_0), (V, V_0))$ is non-trivial only for $m = 0$. In this case the corresponding Lie algebra of vector fields is C5c.

11. I. $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \alpha, y\frac{\partial}{\partial x}, x\frac{\partial}{\partial y} \rangle$.

II. All finite-dimensional submodules have the form:

$$V_m = \langle x^i y^j \mid 0 \leq i + j \leq m \rangle, \quad m \in \mathbb{N} \cup \{0\}.$$

III. The cohomology space $H^2((\mathfrak{g}, \mathfrak{g}_0), (V, V_0))$ is non-zero if and only if $m = 0$ and $\alpha = 2$. The corresponding Lie algebra of vector fields is C6c.

12. I. $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 2x\frac{\partial}{\partial x} + \alpha, x^2\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \alpha x \rangle$.

II. Finite-dimensional submodules exist for any α and have the form:

$$V_{\alpha, m} = \langle x^i e^{(m+\alpha)y} \mid 0 \leq i \leq m \rangle, \quad m \in \mathbb{N} \cup \{0\}.$$

III. Let

$$V = V_{\alpha, m_1} \oplus \cdots \oplus V_{\alpha, m_k}, \quad 0 \leq m_1 < \cdots < m_k.$$

The cohomology space $H^2((\mathfrak{g}, \mathfrak{g}_0), (V, V_0))$ is non-trivial in the following two cases: $\alpha = 0, m_1 > 0$ and $\alpha = 2$. In both cases the cohomology space is one-dimensional, and the corresponding Lie algebras of vector fields are C7c and C7d.

13, $n = 0$. I. $\langle \frac{\partial}{\partial x}, 2x \frac{\partial}{\partial x} - m, x^2 \frac{\partial}{\partial x} - mx, \frac{\partial}{\partial y} \rangle$.

II. Finite-dimensional submodules exist only if $m \in \mathbb{N} \cup \{0\}$ and have the form:

$$V_{p(y)} = \langle x^i f(y) \mid 0 \leq i \leq m, a_0 f + a_1 f' + \dots + a_k f^{(k)} = 0 \rangle,$$

where $p(y) = a_0 + a_1 y + \dots + a_k y^k$, $a_k \neq 0$.

The cohomology space $H^2((\mathfrak{g}, \mathfrak{g}_0), (V, V_0))$ is non-zero only in case $m = 0$. In this case it is one-dimensional, and the corresponding Lie algebra of vector fields is C8c.

17, $n \geq 1$. I. $\langle \frac{\partial}{\partial x}, 2x \frac{\partial}{\partial x} + ny \frac{\partial}{\partial y} - m, x^2 \frac{\partial}{\partial x} + nxy \frac{\partial}{\partial y} - mx, \frac{\partial}{\partial y}, x \frac{\partial}{\partial y}, \dots, x^n \frac{\partial}{\partial y} \rangle$.

II. Finite-dimensional submodules exist only if $m \in \mathbb{N} \cup \{0\}$ and have the form:

$$V_{n,m,p} = \langle x^i y^j \mid 0 \leq j \leq p, 0 \leq i \leq m - jn \rangle, \quad 0 \leq p \leq \left\lceil \frac{m}{n} \right\rceil;$$

The cohomology space $H^2((\mathfrak{g}, \mathfrak{g}_0), (V, V_0))$ is non-trivial only in following two cases: $m = 0$ or $m = kn - 2$. In these cases we have

$$\dim H^2((\mathfrak{g}, \mathfrak{g}_0), (V, V_0)) = \begin{cases} 0, & m \neq 0, kn - 2, k \in \mathbb{N}; \\ 1, & m = 0, n \neq 1, 2; \\ 1, & m = kn - 2, p = k - 2, k \geq 2, n \geq 1, (k, n) \neq (2, 1); \\ 1, & m = kn - 2, p = k - 1, k \geq 1, n \geq 2, (k, n) \neq (1, 2); \\ 2, & m = 0, n = 1, 2. \end{cases}$$

The corresponding Lie algebras of vector fields are listed as C9c-C9g.

18. I. $\langle \frac{\partial}{\partial x}, 2x \frac{\partial}{\partial x} - \alpha, y \frac{\partial}{\partial y} - \beta, x^2 \frac{\partial}{\partial x} + nxy \frac{\partial}{\partial y} - (\alpha + n\beta)x, \frac{\partial}{\partial y}, x \frac{\partial}{\partial y}, \dots, x^n \frac{\partial}{\partial y} \rangle$.

II. Finite-dimensional submodules exist only if $\alpha + n\beta \in \mathbb{N} \cup \{0\}$ and have the form:

$$V_p = \langle x^i y^j \mid 0 \leq j \leq p, 0 \leq i \leq \alpha + n(\beta - j) \rangle, \quad 0 \leq np \leq \alpha + n\beta.$$

III. The cohomology space $H^2((\mathfrak{g}, \mathfrak{g}_0), (V, V_0))$ is non-trivial only in the following cases:

- a) $\alpha = \beta = 0, p = 0, n \geq 0$;
- b) $\alpha = \beta = 0, n = 0, p > 0$;
- c) $\alpha = 0, n = 0, \beta = p + 1$;
- d) $\alpha = -2, \beta = k \geq 2, p = k - 2$;
- e) $\alpha = -2, \beta = k \geq 1, p = k - 1$.

In all these cases the cohomology space is one-dimensional, and the corresponding Lie algebras of vector fields are C10c-C10g.

5. REAL CASE

5.1. Homogeneous spaces without one-dimensional invariant distribution. Over the field of real numbers we have the following primitive pairs of codimension 3:

- (1) $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{R}), \mathfrak{g}_0$ is a parabolic subalgebra of codimension 3;
- (2) $\mathfrak{g} = \mathfrak{gl}(3, \mathbb{R}) \ltimes \mathbb{R}^3, \mathfrak{g}_0 = \mathfrak{gl}(3, R)$;
- (3) $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R}) \ltimes \mathbb{R}^3, \mathfrak{g}_0 = \mathfrak{sl}(3, R)$;

- (4a) $\mathfrak{g} = \mathfrak{so}(4, 1)$, \mathfrak{g}_0 is a parabolic subalgebra of codimension 3;
- (4b) $\mathfrak{g} = \mathfrak{so}(3, 2)$, \mathfrak{g}_0 is a parabolic subalgebra of codimension 3;
- (5a) $\mathfrak{g} = \mathfrak{co}(3) \ltimes \mathbb{R}^3$, $\mathfrak{g} = \mathfrak{co}(3)$;
- (5b) $\mathfrak{g} = \mathfrak{co}(2, 1) \ltimes \mathbb{R}^3$, $\mathfrak{g} = \mathfrak{co}(2, 1)$;
- (6a) $\mathfrak{g} = \mathfrak{so}(4)$, $\mathfrak{g}_0 = \mathfrak{so}(3)$;
- (6b) $\mathfrak{g} = \mathfrak{so}(3, 1)$, $\mathfrak{g}_0 = \mathfrak{so}(3)$;
- (6c) $\mathfrak{g} = \mathfrak{so}(3, 1)$, $\mathfrak{g}_0 = \mathfrak{so}(2, 1)$;
- (6d) $\mathfrak{g} = \mathfrak{so}(2, 2)$, $\mathfrak{g}_0 = \mathfrak{so}(2, 1)$;
- (7a) $\mathfrak{g} = \mathfrak{so}(3) \ltimes \mathbb{R}^3$, $\mathfrak{g}_0 = \mathfrak{so}(3)$;
- (7b) $\mathfrak{g} = \mathfrak{so}(2, 1) \ltimes \mathbb{R}^3$, $\mathfrak{g}_0 = \mathfrak{so}(2, 1)$;
- (8) $\mathfrak{g} = \mathfrak{sp}(4, \mathbb{R})$, \mathfrak{g}_0 is a parabolic subalgebra of codimension 3;
- (13b') $\mathfrak{g} = \mathfrak{su}(2, 1) = \left\{ \begin{pmatrix} z_1 & z_2 & ix \\ z_3 & \bar{z}_1 - z_1 - \bar{z}_2 & -\bar{z}_1 \\ iy & -\bar{z}_3 & -\bar{z}_1 \end{pmatrix} \mid z_1, z_2, z_3 \in \mathbb{C}, x, y \in \mathbb{R} \right\},$
 $\mathfrak{g}_0 = \left\{ \begin{pmatrix} z_1 & z_2 & ix \\ 0 & \bar{z}_1 - z_1 - \bar{z}_2 & -\bar{z}_1 \\ 0 & 0 & -\bar{z}_1 \end{pmatrix} \right\},$

The pair (13b') is equivalent over \mathbb{C} to the pair (13b) given in Subsection 4.2.

We have three additional non-solvable Lie algebras of vector fields on the real plane, which are listed in Subsection A.2 under numbers 7', 7'' and 8'. Notice that all these Lie algebras are simple. Namely, Lie algebra 7' is isomorphic to $\mathfrak{su}(2)$, 7'' is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$, and 8' is isomorphic to $\mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}$.

The following lemma helps to describe all real imprimitive effective pairs $(\mathfrak{g}, \mathfrak{g}_0)$ of codimension 3 such that the isotropy \mathfrak{g}_0 -module $\mathfrak{g}/\mathfrak{g}_0$ does not have one-dimensional submodules.

Lemma 7. *Let $(\mathfrak{g}, \mathfrak{g}_0)$ be an effective pair of Lie algebras of arbitrary codimension, let \mathfrak{h} be a subalgebra of codimension 1 in \mathfrak{g} , containing \mathfrak{g}_0 , and let \mathfrak{b}_0 be the maximal ideal of \mathfrak{h} lying in \mathfrak{g}_0 . Suppose that the Lie algebra $\mathfrak{h}/\mathfrak{b}_0$ is simple. Then the \mathfrak{g}_0 -module $\mathfrak{g}/\mathfrak{g}_0$ has a one-dimensional submodule complementary to $\mathfrak{h}/\mathfrak{g}_0$.*

Proof. Let \mathfrak{a} be a simple subalgebra in \mathfrak{h} , complementary to \mathfrak{b}_0 . This subalgebra always exists by the Levy theorem. Since \mathfrak{a} is simple, the \mathfrak{a} -module \mathfrak{g} is also simple and there exists a one-dimensional trivial submodule $V \subset \mathfrak{g}$ complementary to \mathfrak{h} .

Let v be a non-zero vector in V . Then, clearly, we have $\text{ad } v(V + \mathfrak{a}) = \{0\}$. Define the decreasing sequence of subspaces in \mathfrak{g} as follows:

$$\mathfrak{b}_i = \{x \in \mathfrak{b}_{i-1} \mid [v, x] \in V + \mathfrak{b}_{i-1} \text{ for all } i \geq 1\}.$$

Let us prove by induction by i that each subspace \mathfrak{b}_i is a submodule of the \mathfrak{a} -module \mathfrak{b}_0 . Indeed, for $i = 0$ we have nothing to prove. Suppose that $i \geq 1$ and \mathfrak{b}_{i-1} is a submodule of the \mathfrak{a} -module \mathfrak{b}_0 . Then for any $x \in \mathfrak{a}$, $y \in \mathfrak{b}_i$ we have $[x, y] \in \mathfrak{b}_{i-1}$ and

$$\text{ad } v([x, y]) = [x, \text{ad } v(y)] \subset [x, V + \mathfrak{b}_{i-1}] \subset \mathfrak{b}_{i-1}.$$

Hence, $[x, y] \in \mathfrak{b}_i$ and \mathfrak{b}_i is stable with respect to the action of \mathfrak{a} .

Let n be the smallest integer such that $\mathfrak{b}_n = \mathfrak{b}_{n+1}$. If $n = 0$, we get

$$[V, \mathfrak{g}_0] \subset [V, \mathfrak{b}_0] \subset V + \mathfrak{b}_0 \subset V + \mathfrak{g}_0.$$

This means that $V + \mathfrak{g}_0$ is a one-dimensional \mathfrak{g}_0 -submodule of $\mathfrak{g}/\mathfrak{g}_0$ complementary to $\mathfrak{h}/\mathfrak{g}_0$.

Suppose that $n \geq 1$. Since the \mathfrak{a} -module \mathfrak{b}_{n-1} is semisimple, we see that there exists a submodule $W_{n-1} \subset \mathfrak{b}_{n-1}$ complementary to \mathfrak{b}_n . Define $W_i = \text{ad } v(W_{i+1})$ for all $i = 0, \dots, n-2$. Since $\text{ad } v$ is a homomorphism of \mathfrak{a} -modules we can easily show that $\mathfrak{b}_i = W_i \oplus \mathfrak{b}_{i+1}$ for all $i = 0, \dots, n-1$ (direct sum of \mathfrak{a} -modules) and $\text{ad } v(W_0) = \mathfrak{a}$. Let us define for convenience W_{-1} to be \mathfrak{a} .

Finally, by induction by i we can prove that $\text{ad } v([W_0, W_i]) = W_i$ which for $i = n-1$ will take form $\text{ad } v(W_0, W_{n-1}) = W_{n-1}$. But this is impossible, since

$$\text{ad } v(\mathfrak{g}) = \text{ad } v(V \oplus W_{-1} \oplus W_0 \oplus \dots \oplus W_{n-1} \oplus \mathfrak{b}_n) \subset V \oplus W_{-1} \oplus W_0 \oplus \dots \oplus W_{n-2} \oplus \mathfrak{b}_n,$$

so that W_{n-1} does not belong to the image of $\text{ad } v$. \square

Suppose now that $(\mathfrak{g}, \mathfrak{g}_0)$ is a real imprimitive effective pair of codimension 3 such that the isotropy \mathfrak{g}_0 -module $\mathfrak{g}/\mathfrak{g}_0$ does not contain one-dimensional submodules. Nevertheless, since the pair $(\mathfrak{g}, \mathfrak{g}_0)$ is not primitive, there exist a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ containing \mathfrak{g}_0 such that $\text{codim}_{\mathfrak{g}} \mathfrak{h} = 1$. Let \mathfrak{b}_0 be the maximal ideal of \mathfrak{h} that lies in \mathfrak{g}_0 . Then the pair $(\mathfrak{h}/\mathfrak{b}_0, \mathfrak{g}_0/\mathfrak{b}_0)$ is effective and primitive. Moreover, in view of Lemma 7 the Lie algebra $\mathfrak{h}/\mathfrak{b}_0$ can not be simple. Hence, the pair $(\mathfrak{h}/\mathfrak{b}_0, \mathfrak{g}_0/\mathfrak{b}_0)$ corresponds to one of the two Lie algebras of vector fields on the plane under numbers 10 or 11. These two cases were considered over \mathbb{C} by Sophus Lie in [7]. Tracing his consideration of these two cases, we see that it goes without any changes over the field of real numbers. Hence, over \mathbb{R} we get the same list of Lie algebras on \mathbb{R}^3 , which are given in Subsection A.4.

5.2. Homogeneous spaces with one-dimensional invariant distribution. Below we describe additional Lie algebras of vector fields on the plane with an invariant one-dimensional distribution, that appear if we apply methods of Subsection 4.3 to real Lie algebras $7'$, $7''$, and $8'$.

5.2.1. Subalgebras of codimension 1 in stationary subalgebras. Let g be one of the Lie algebras $7'$, $7''$, or $8'$, and let \tilde{g} be its stationary subalgebra. As in Subsection 4.2, we list now all subalgebras \mathfrak{g}_0 in \mathfrak{h}_0 of codimension 1, such that there are no any ideals \mathfrak{a} in \mathfrak{g} , which satisfy $\text{codim}_{\mathfrak{a}+\mathfrak{g}_0} \mathfrak{g}_0 = 1$. All subalgebras with this property have the form:

$$7': \mathfrak{g} = \mathfrak{su}(2), \mathfrak{g}_0 = \{0\};$$

$$7'': \text{see [6] and [7] in Subsection 4.2.}$$

$$8': \mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})_{\mathbb{R}}, \mathfrak{g}_0 = \left\{ \begin{pmatrix} e^{i\alpha x} & z \\ 0 & -e^{i\alpha x} \end{pmatrix} \mid x \in \mathbb{R}, z \in \mathbb{C} \right\} (\alpha \sim -\alpha).$$

The corresponding Lie algebras of vector fields are $B1'$ and $B2'$ from Subsection A.7.

5.2.2. Extensions of real pairs. Below we consider extensions of the pairs $7'$, $7''$, and $8'$ in the same way as in Subsection 4.3.

7'. I. $\langle x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} + \alpha, (1 + x^2 - y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} + 2\alpha y, 2xy \frac{\partial}{\partial x} + (1 - x^2 + y^2) \frac{\partial}{\partial y} - 2\alpha x \rangle$.

II. Finite-dimensional submodules exist only if $\alpha = 0$. In this case for each $n \in \mathbb{N} \cup \{0\}$ there exists a unique submodule V_n of dimension $2n + 1$, which is generated (as a submodule) by $P_n(\frac{x^2+y^2-1}{x^2+y^2+1})$, where $P_n(z)$ is the n -th Legendre polynomial:

$$P_n(z) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dz^n} (1 - z^2)^n.$$

III. The cohomology space $H^2((\mathfrak{g}, \mathfrak{g}_0), (V, V_0))$ is always one-dimensional, and the corresponding Lie algebras of vector fields are C2c' (with sign +).

Remark 1. This Lie algebra of vector fields corresponds to the classical action of the Lie group $SO(3)$ on S^2 . In spherical coordinates it has the form:

$$\left\langle \frac{\partial}{\partial y}, \sin(y) \frac{\partial}{\partial x} + \cot(x) \cos(y) \frac{\partial}{\partial y}, \cos(y) \frac{\partial}{\partial x} - \cot(x) \sin(y) \frac{\partial}{\partial y} \right\rangle.$$

Then the modules V_n can be written explicitly in terms of adjoint Legendre functions:

$$P_l^m(z) = \frac{(-1)^{m+l} (1 - z^2)^{m/2}}{2^l l!} \frac{d^{m+l}}{dz^{m+l}} (1 - z^2)^l.$$

Namely, we have

$$V_n = \langle P_n(\cos(x)), \sin(ky) P_n^k(\cos(x)), \cos(ky) P_n^k(\cos(x)) \mid k = 1, \dots, n \rangle.$$

7''. I. $\langle x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} + \alpha, (1 - x^2 + y^2) \frac{\partial}{\partial x} - 2xy \frac{\partial}{\partial y} - 2\alpha y, -2xy \frac{\partial}{\partial x} + (1 + x^2 - y^2) \frac{\partial}{\partial y} + 2\alpha x \rangle$.

II. Finite-dimensional submodules exist only if $\alpha = 0$. In this case for each $n \in \mathbb{N} \cup \{0\}$ there exists a unique submodule V_n of dimension $2n + 1$, which is generated (as a submodule) by $P_n(\frac{x^2+y^2+1}{x^2+y^2-1})$, where $P_n(z)$ is the n -th Legendre polynomial.

III. The cohomology space $H^2((\mathfrak{g}, \mathfrak{g}_0), (V, V_0))$ is always one-dimensional, and the corresponding Lie algebras of vector fields are C2c' (with sign -).

Remark 2. This Lie algebra of vector fields corresponds to the classical action of the Lie group $PSL(2, \mathbb{R})$ on the hyperbolic plane H^2 . As above, we could introduce an analog of spherical coordinates on H^2 and would get a representation of submodules V_n in terms of adjoint Legendre functions. (We would need only to replace $\cot(x)$ to $\coth(x)$ in Remark 1.) Instead of that we give an explicit representation of submodules V_n in Poincare model of H^2 . In this model H^2 is identified with the set of all complex numbers $z = x + iy$ with positive imaginary part, and $PSL(2, \mathbb{R})$ acts by projective transformations on z . The corresponding Lie algebra of vector fields has the form:

$$\left\langle \frac{\partial}{\partial x}, x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}, (x^2 - y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} \right\rangle.$$

Then the modules V_n take the form:

$$V_n = \langle \frac{\partial^k}{\partial x^k} \frac{(x^2+y^2)^n}{y^n} \mid k = 0, \dots, 2n \rangle.$$

8'. I. $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} - \alpha, x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} - \beta, (x^2 - y^2) \frac{\partial}{\partial x} + 2xy \frac{\partial}{\partial y} - 2(\alpha x - \beta y), -2xy \frac{\partial}{\partial x} + (x^2 - y^2) \frac{\partial}{\partial y} + 2(\alpha y + \beta x) \rangle$.

II. Finite-dimensional submodules exist only if $\alpha \in \mathbb{N} \cup \{0\}$ and $\beta = 0$. In this case it is unique and has the form:

$$V = \{x^i y^j (x^2 + y^2)^k \mid i + j + k \leq \alpha\}.$$

III. The cohomology space $H^2((\mathfrak{g}, \mathfrak{g}_0), (V, V_0))$ is non-trivial only if $\alpha = 0$. In this case it is two-dimensional, and the corresponding Lie algebra is given by C3c'.

6. COMPUTATION OF COHOMOLOGY IN CASE 13

We can assume that

$$\begin{aligned} \mathfrak{sl}(2, \mathbb{C}) &= \langle \frac{\partial}{\partial x}, 2x \frac{\partial}{\partial x} + ny \frac{\partial}{\partial y} + mz \frac{\partial}{\partial z}, x^2 \frac{\partial}{\partial x} + nxy \frac{\partial}{\partial y} + mxz \frac{\partial}{\partial z} \rangle, \\ V &= \langle \frac{\partial}{\partial y}, x \frac{\partial}{\partial y}, \dots, \frac{x^n}{n!} \frac{\partial}{\partial y} \rangle, \\ W_p &= \langle y^p \frac{\partial}{\partial z}, xy^p \frac{\partial}{\partial z}, \dots, \frac{x^{m-np} y^p}{(m-np)!} \frac{\partial}{\partial z} \rangle, \end{aligned}$$

where $1 \leq p \leq [m/n]$.

Let us describe all mappings

$$\alpha: V \wedge V \rightarrow W_p,$$

such that the space

$$\mathfrak{sl}(2, \mathbb{C}) \times V \times (W_p \times W_{p-1} \times \dots \times W_0)$$

with the multiplication $[v_1, v_2] = \alpha(v_1, v_2)$ for all $v_1, v_2 \in V$ and natural multiplication on all other summands will be a Lie algebra.

This implies necessary and sufficient conditions on α :

- (1) α is an $\mathfrak{sl}(2, \mathbb{C})$ -invariant mapping;
- (2) $\{[v_1, [v_2, v_3]]\} = 0$ for all $v_1, v_2, v_3 \in V$, that is α is a 2-cocycle of the V -module $W_p \times \dots \times W_0$.

Assume that

$$\alpha \left(\frac{x^i}{i!} \frac{\partial}{\partial y}, \frac{x^j}{j!} \frac{\partial}{\partial y} \right) = \sum_{k=0}^{m-pn} a_k^{ij} \frac{x^k y^p}{k!} \frac{\partial}{\partial z}.$$

Then from the invariance of α with respect to the action of $2x \frac{\partial}{\partial x} + ny \frac{\partial}{\partial y} + mz \frac{\partial}{\partial z} \in \mathfrak{sl}(2, \mathbb{C})$ we get that

$$\left(k - i - j - \frac{m - (n+2)p}{2} \right) a_k^{ij} = 0.$$

Hence, if $k \neq i + j + \frac{m - (n+2)p}{2}$, then $a_k^{ij} = 0$, or, what is the same,

$$\alpha \left(\frac{x^i}{i!} \frac{\partial}{\partial y}, \frac{x^j}{j!} \frac{\partial}{\partial y} \right) = a_k^{ij} \frac{x^k y^p}{k!} \frac{\partial}{\partial z}, \quad \text{where } k = i + j + \frac{m - (n+2)p}{2}.$$

From the Jacobi identity on elements $\frac{\partial}{\partial y}$, $\frac{x^i}{i!} \frac{\partial}{\partial y}$, and $\frac{x^j}{j!} \frac{\partial}{\partial y}$ we have

$$\left[\frac{\partial}{\partial y}, \alpha \left(\frac{x^i}{i!} \frac{\partial}{\partial y}, \frac{x^j}{j!} \frac{\partial}{\partial y} \right) \right] = \left[\frac{x^i}{i!} \frac{\partial}{\partial y}, \alpha \left(\frac{\partial}{\partial y}, \frac{x^j}{j!} \frac{\partial}{\partial y} \right) \right] - \left[\frac{x^j}{j!} \frac{\partial}{\partial y}, \alpha \left(\frac{\partial}{\partial y}, \frac{x^i}{i!} \frac{\partial}{\partial y} \right) \right],$$

or explicitly,

$$p \sum_{k=0}^{m-pn} a_k^{ij} \frac{x^k y^{p-1}}{k!} \frac{\partial}{\partial z} = \frac{p}{i!} \sum_{k=0}^{m-pn} a_k^{0j} \frac{x^{k+i} y^{p-1}}{k!} \frac{\partial}{\partial z} - \frac{p}{j!} \sum_{k=0}^{m-pn} a_k^{0i} \frac{x^{k+j} y^{p-1}}{k!} \frac{\partial}{\partial z}.$$

This implies that

$$(3) \quad a_k^{ij} = C_k^i a_{k-i}^{0j} - C_k^j a_{k-j}^{0i},$$

where we assume that $a_k^{ij} = 0$ for negative k .

From the invariance of α with respect to $\frac{\partial}{\partial x} \in \mathfrak{sl}(2, \mathbb{C})$ we see that

$$\left[\frac{\partial}{\partial x}, \alpha \left(\frac{\partial}{\partial y}, \frac{x^i}{i!} \frac{\partial}{\partial y} \right) \right] = \alpha \left(\frac{\partial}{\partial y}, \frac{x^{i-1}}{(i-1)!} \frac{\partial}{\partial y} \right), \quad \text{for all } 1 \leq i \leq n.$$

Therefore,

$$\sum_{k=1}^{m-pn} a_k^{0i} \frac{x^{k-1} y^p}{(k-1)!} \frac{\partial}{\partial z} = \sum_{k=0}^{m-pn} a_k^{0, i-1} \frac{x^k y^p}{k!} \frac{\partial}{\partial z},$$

or

$$\sum_{k=1}^{m-pn-1} a_{k+1}^{0i} \frac{x^k y^p}{k!} \frac{\partial}{\partial z} = \sum_{k=0}^{m-pn} a_k^{0, i-1} \frac{x^k y^p}{k!} \frac{\partial}{\partial z}.$$

Since $a_k^{0,0} = 0$ for all k , we obtain

$$\begin{aligned} a_k^{0,1} &= 0 \quad \text{for all } 1 \leq k \leq m-np, \\ a_{m-pn}^{0,i} &= 0 \quad \text{for all } 0 \leq i \leq n-1, \\ a_{k+1}^{0,i} &= a_k^{0, i-1} \quad \text{for all } 1 \leq i \leq n, \quad 0 \leq k \leq m-np-1. \end{aligned}$$

In particular, this implies that

$$a_k^{0,i} = \begin{cases} 0 & \text{for } k > i, \\ a_0^{0, i-k} & \text{for } k < i, \end{cases}$$

and the equality (3) takes the form

$$a_k^{i,j} = \begin{cases} 0 & \text{for } k \geq i+j, \\ (C_k^i - C_k^j) a_0^{0, i+j-k} & \text{for } k < i+j. \end{cases}$$

Finally, from the invariance of α with respect to $x^2 \frac{\partial}{\partial x} + nxy \frac{\partial}{\partial y} + mxz \frac{\partial}{\partial z} \in \mathfrak{sl}(2, \mathbb{C})$ we get

$$\left[x^2 \frac{\partial}{\partial x} + nxy \frac{\partial}{\partial y} + mxz \frac{\partial}{\partial z}, \alpha \left(\frac{\partial}{\partial y}, x \frac{\partial}{\partial y} \right) \right] = \alpha \left(\frac{\partial}{\partial y}, (1-n)x^2 \frac{\partial}{\partial y} \right).$$

Let $k = 1 + \frac{m-(n+2)p}{2}$. Then this equality takes the form:

$$\left[x^2 \frac{\partial}{\partial x} + nxy \frac{\partial}{\partial y} + mxz \frac{\partial}{\partial z}, (C_k^0 - C_k^1) a_0^{0,1-k} \frac{x^k y^p}{k!} \frac{\partial}{\partial z} \right] = 2(1-n)(C_{k+1}^0 - C_{k+1}^2) a_0^{0,1-k} \frac{x^{k+1} y^p}{(k+1)!} \frac{\partial}{\partial z}.$$

If $a_0^{0,1-k} = 0$ then it is easy to see that the mapping α is trivial. Assume that $a_0^{0,1-k} \neq 0$. Then

$$(k+1)(1-k)(k+np-m) = 2(1-n)(1-k(k+1)/2).$$

Let $x = \frac{m-np}{2}$. Then $k = 1 - n + x$ and after simple manipulation we obtain the following second order equation on x :

$$(2-n+x)(1-n-x) = (1-n)(3-n+x).$$

This equation has two roots $x = -1$ and $x = n-1$. The first solution does not have sense, since $\dim W = m-np+1 = 2x+1 > 0$. Therefore, non-zero mapping α may exist only if $m-pn = 2n-2$. In this case we can put $a_0^{0,1} = 1$ and obtain

$$\alpha \left(\frac{x^i}{i!} \frac{\partial}{\partial y}, \frac{x^j}{j!} \frac{\partial}{\partial y} \right) = (C_{i+j-1}^i - C_{i+j-1}^j) \frac{x^k y^p}{k!} \frac{\partial}{\partial z}.$$

7. ONE CLASS OF NILPOTENT TRANSFORMATION GROUPS

Let \mathfrak{g} be an arbitrary finite-dimensional (real or complex) Lie algebra. Then the pair $(\mathfrak{g}, \{0\})$ is obviously effective. Consider an effective $(\mathfrak{g}, \{0\})$ -module (V, V_0) of codimension 1. This means that V is an arbitrary \mathfrak{g} -module, V_0 is any subspace of codimension 1 in V , and V_0 contains no non-zero submodules of the \mathfrak{g} -module V . From Theorem ?? it follows that these modules are in one-to-one correspondence with left ideals J of finite codimension in the universal enveloping algebra $U(\mathfrak{g})$. If J is an ideal of this kind, then $V = (U(\mathfrak{g})/J)^*$ and V_0 is the kernel of the element $1+J$ viewed as a linear form on V . So, we can associate the effective pair $(\mathfrak{g} \ltimes V, V_0)$ to each left ideal J of finite codimension in $U(\mathfrak{g})$, whose codimension is one greater than the dimension of \mathfrak{g} .

The corresponding homogeneous space can be locally described in the following way. Let G be a Lie group whose Lie algebra is isomorphic to \mathfrak{g} . Then J can be regarded as the set of left-invariant differential operators on \mathfrak{g} . Hence it defines the following subspace \mathcal{F} in the algebra $C^{\text{inf}}(G)$ of all smooth functions on G (or, in the algebra $C^\omega(G)$ of all analytical functions for the complex case):

$$\mathcal{F} = \{f \in C^{\text{inf}}(G) \mid Df = 0 \text{ for all } D \in J\}.$$

Since J has finite codimension, it is easy to see that this subspace is finite-dimensional. Since J is a left ideal in $U(\mathfrak{g})$, the space \mathcal{F} will be stable under the natural action of G on $C^{\text{inf}}(G)$: $(g.f)(h) = f(g^{-1}h)$ for all $g, h \in G$. This supplies \mathcal{F} with the structure of a G -module. Define the action of the Lie group $G \ltimes \mathcal{F}$ on the manifold $G \times \mathbb{C}$ as

$$(g, f).(h, a) = (gh, f(gh) + a), \quad g, h \in G, f \in \mathcal{F}, a \in \mathbb{C}.$$

It is easy to see that this action is effective. If $J \neq U(\mathfrak{g})$, then $\mathcal{F} \neq \{0\}$ and $G \ltimes \mathcal{F}$ acts transitively on $G \times \mathbb{C}$. The stationary subgroup of this action at the point $(e, 0)$ is $\{e\} \times \mathcal{F}_0$ where $\mathcal{F}_0 = \{f \in \mathcal{F} \mid f(e) = 0\}$.

So, we see that the problem of description of all homogeneous spaces in dimension $n + 1$ includes, in particular, the description of all left ideals of finite codimension in $U(\mathfrak{g})$ for all n -dimensional Lie algebras \mathfrak{g} .

Consider, for example, the case when $\mathfrak{g} = \mathbb{C}^n$ is a commutative Lie algebra. Then $U(\mathfrak{g}) = \mathbb{C}[x_1, \dots, x_n]$, and one needs to describe all ideals of finite codimension in the algebra of all polynomials in n variables. For $n = 1$ this problem is straightforward, since all ideals are principal. But it is no longer the case for $n > 1$, and the problem of describing the ideals of finite codimension becomes considerably more difficult. For example, it was investigated by Zarisky[10] for the case $n = 2$ in connection with singularities of plane curves, and also in [8], where the description of all so called complete ideals in $\mathbb{C}[x_1, x_2]$ is presented. And one of the discrete invariants of these ideals is the set of finite sequences of positive rational numbers.

REFERENCES

- [1] Boris Doubrov, *One-dimensional invariant distributions on homogeneous spaces*, Dokl. Acad. Sci. Belarus, v. 42, no. 3, 1997, 26–30.
- [2] Kim Sen En, Morozov V.V., *On imprimitive groups of three-dimensional complex space*, Uchenye zapiski, Kazan State University, v. 115, book 14, 1965, 69–85 (in Russian).
- [3] D. Fuks, *Chomology of infinite-dimensional Lie algebras*, Contemporary Soviet Mathematics. New York: Consultants Bureau, 1986.
- [4] R. Milson, *Representations of finite-dimensional Lie algebras by first-order differential operators. Some local results in the transitive case*, J. Lond. Math. Soc., II. Ser. textbf52, 1995, 285–302.
- [5] A. González-López, N. Kamran, P. Olver, *Lie algebras of differential operators in two complex variables*, Am. J. Math., **114**, 1992, 1163–1185.
- [6] Sophus Lie Archive, package XXI (located at the Norwegian State Library).
- [7] S. Lie, *Theorie der Transformationsgruppen*, Bd. 3, Teubner, Leipzig, 1893.
- [8] M. Spivakovsky. *Valuations in function fields of surfaces*. Amer. J. Math., **112** (1990), 107–156.
- [9] E. Stitzinger, *On the Frattini subalgebra of a Lie algebra*, J. Lond. Math. Soc., II. Ser. **2**, 1970, 429–438.
- [10] O. Zariski. *Polynomial ideals defined by infinitely near base points*. Amer. J. Math., **60** (1938), 151–204.

APPENDIX A. SUMMARY OF RESULTS

A.1. Notation.

- $p = \frac{\partial}{\partial x}$, $q = \frac{\partial}{\partial y}$, $r = \frac{\partial}{\partial z}$;
- $V_{p(t)} = \{f(x) \mid a_n f^{(n)}(x) + \dots + a_1 f'(x) + a_0 f(x) = 0\}$, for any polynomial $p(t) = a_n t^n + \dots + a_1 t + a_0$;
- $V_{\alpha, m} = \langle x^i e^{(m-\alpha)} \mid 0 \leq i \leq m \rangle$ and $V_m = V_{0, m}$;

A.2. Transitive Lie algebras of vector fields on the plane.

Solvable algebras

1. $\langle p, f(x)q \mid f \in V_{p(t)} \rangle$, $\deg p \geq 1$, $p(t) \sim p(at)$ for all $a \in \mathbb{C}^*$;
2. $\langle p, yq, f(x)q \mid f \in V_{p(t)} \rangle$, $\deg p \geq 1$, $p(t) \sim p(at + b)$ for all $a \in \mathbb{C}^*$, $b \in \mathbb{C}$;

3. $\langle p, xp + \alpha yq, q, xq, \dots, x^n q \rangle$, $n \geq 0$; if $n = 0$ then $\alpha \sim 1/\alpha$;
4. $\langle p, xp + (n+1)yq + x^{n+1}q, q, xq, \dots, x^n q \rangle$, $n \geq 0$;
5. $\langle p, xp, yq, q, xq, \dots, x^n q \rangle$, $n \geq 0$.

Non-solvable algebras

6. $\langle p, 2xp - q, x^2p - xq \rangle$.
7. $\langle p, xp - yq, x^2p - (1 + 2xy)q \rangle$.
8. $\langle p, q, xp, yq, x^2p, y^2q \rangle$.
9. $\langle p, q, xp, xq, yp, yq, x^2p + xyq, xyp + y^2q \rangle$.
10. $\langle p, q, xp - yq, xq, yp \rangle$.
11. $\langle p, q, xp, xq, yq, yp \rangle$.
12. $\langle p, q, xp, x^2p - xq \rangle$.
13. $\langle p, 2xp + nyq, x^2p + nxyq, q, xq, \dots, x^n q \rangle$, $n \geq 0$.
14. $\langle p, xp, yq, x^2p + nxyq, q, xq, \dots, x^n q \rangle$, $n \geq 0$.

Additional non-solvable algebras in real case

- 7'. $\langle xq - yp, (1 + x^2 - y^2)p + 2xyq, 2xyp + (1 - x^2 + y^2)q \rangle$.
- 7''. $\langle xq - yp, (1 - x^2 + y^2)p - 2xyq, -2xyp + (1 + x^2 - y^2)q \rangle$.
- 8'. $\langle p, q, xp + yq, xq - yp, (x^2 - y^2)p + 2xyq, -2xyp + (x^2 - y^2)q \rangle$.

A.3. Primitive Lie algebras in space.

- P1. $\langle p, q, r, xp, xq, xr, yp, yq, yr, zp, zq, zr, xU, yU, zU \rangle$, $U = xp + yq + zr$;
- P2. $\langle p, q, r, xp, xq, xr, yp, yq, yr, zp, zq, zr \rangle$;
- P3. $\langle p, q, r, xp - zr, xq, xr, yp, yq - zr, yr, zp, zq \rangle$;
- P4. $\langle p, q, r, xq - yp, xr - zp, yr - zq, xp + yq + zr, 2xU - Sp, 2yU - Sq, 2zU - Sr \rangle$, $U = xp + yq + zr$, $S = x^2 + y^2 + z^2$;
- P5. $\langle p, q, r, xq - yp, xr - zp, yr - zq, xp + yq + zr \rangle$;
- P6. $\langle xq - yp, xr - zp, yr - zq, Sp + 2xU, Sq + 2yU, Sr + 2zU \rangle$, where $S = 1 - x^2 - y^2 - z^2$, $U = xp + yq + zr$;
- P7. $\langle p, q, r, xq - yp, xr - zp, yr - zq \rangle$;
- P8. $\langle 2p - yr, 2q + xr, r, xp - yq, xq, yp, xp + yq + 2zr, x(xp + zr) + (xy + 2z)q, (xy - 2z)p + y(yp + zr), z(xp + yq + zr) \rangle$.

A.4. Imprimitive Lie algebras without one-dimensional invariant foliations.

- A1. $\langle p, yq - zr, yr, zq, f(x)q, f(x)r \mid f \in V(p) \rangle$;
- A2. $\langle p, yq, yr, zq, zr, f(x)q, f(x)r \mid f \in V(p) \rangle$;
- A3. $\langle p, xp + \alpha(yq + zr), yq - zr, yr, zq, x^i p, x^i q \mid i = 0, \dots, n \rangle$;
- A4. $\langle p, xp, yq, yr, zq, zr, x^i p, x^i q \mid i = 0, \dots, n \rangle$;
- A5. $\langle p, 2xp + n(yq + zr), x^2p + nx(yq + zr), yq - zr, yr, zq, x^i p, x^i q \mid i = 0, \dots, n \rangle$;
- A6. $\langle p, xp, x^2p + nx(yq + zr), yq, yr, zq, zr, x^i p, x^i q \mid i = 0, \dots, n \rangle$;

A.5. Lie algebras with one-dimensional invariant foliations of type B.

- B1. $\langle p, 2p - q, x^2p - xq + e^{-2y}r \rangle$;
- B2. $\langle p, 2xp + r, x^2p + xr, q, 2yq - \alpha r, y^2q - \alpha yr \rangle$, $\alpha \neq 0$, $\alpha \sim 1/\alpha$;
- B3. $\langle p, q, r + xp + 2yq, xp - yq, yp, xq, x^2p + xyq + xr, xyp + y^2q + yr \rangle$;
- B4. $\langle p, q, r - xq, 2xp + yq - zr, xp - yq - 2zr, x^2p + xyq + (xz + y)r, xyp + y^2q + z(y + xz)r, yp + z^2r \rangle$;
- B5. $\langle p, q, xq, xp - yq, yp, xp + yq + r \rangle$;

- B6. $\langle p, 2xp + nyq + (n-2)zr, x^2p + nxyq + ((n-2)xz + ny)r, q, xq + r, x^2q + 2xr, \dots, x^nq + nx^{n-1}r \rangle, n \geq 1;$
 B7. $\langle p, xp + yq + r, x^2p + (x+2y)r, q, xq + r, x^2q + 2xr \rangle;$
 B8. $\langle p, xp - zr, yq + zr, x^2p + nxyq + ((n-2)xz + ny)r, q, xq + r, x^2q + 2xr, \dots, x^nq + nx^{n-1}r \rangle, n \geq 1.$

A.6. Lie algebras with one-dimensional invariant foliations of types C1–C3.

- C1a. $\langle p, 2xp - q, x^2p - xq, f(x, y)r \mid f \in V_{m_1} + \dots + V_{m_k} \rangle, 0 \leq m_1 < \dots, m_k, \text{ where } V_m = \langle x^i e^{my} \mid 0 \leq i \leq m \rangle;$
 C1b. $\langle p, 2xp - q, x^2p - xq, zr, f(x, y)r \mid f \in V_{m_1} + \dots + V_{m_k} \rangle, 0 \leq m_1 < \dots, m_k;$
 C1c. $\langle p, 2xp - q, x^2p - xq + e^{-2y}r, f(x, y)r \mid f \in V_{m_1} \oplus \dots \oplus V_{m_k} \rangle, 0 \leq m_1 < \dots < m_k;$
 C2a. $\langle p, 2xp - 2yq + nzs, x^2p - (1+2xy)q + nxzs, f(x, y)r \mid f \in V_{n, m_1} + \dots + V_{n, m_k} \rangle, n \geq 0, 0 \leq m_1 < \dots, m_k, \text{ where } V_{n, m} = \langle \frac{d^i}{dx^i} (x^{n+m}(1+xy)^m) \mid 0 \leq i \leq n+2m \rangle;$
 C2b. $\langle p, xp - yq, zr, x^2p - (1+2xy)q + nxzs, f(x, y)r \mid f \in V_{n, m_1} + \dots + V_{n, m_k} \rangle, n \geq 0, 0 \leq m_1 < \dots, m_k;$
 C2c. $\langle p, xp - yq + r, x^2p - (1+2xy)q + 2xr, f(x, y)r \mid f \in V_{0, m_1} + \dots + V_{0, m_k} \rangle, 0 \leq m_1 < \dots, m_k;$
 C3a. $\langle p, 2xp + nzs, x^2p + nxzs, q, 2yq + mzs, y^2q + myzs, x^i y^j \mid 0 \leq i \leq n, 0 \leq j \leq m \rangle, n, m \geq 0;$
 C3b. $\langle p, xp, x^2p + nxzs, q, yq, y^2q + myzs, zr, x^i y^j \mid 0 \leq i \leq n, 0 \leq j \leq m \rangle, n, m \geq 0;$
 C3c. $\langle p, xp, x^2p + xr, q, yq, y^2q + yr, r \rangle;$
 C4a. $\langle p, q, xp + yq + (2n/3)zs, xp - yq, yp, xq, x(xp + yq + nzs), y(xp + yq + nzs), x^i y^j r \mid 0 \leq i+j \leq n, i, j \geq 0 \rangle, n \geq 0;$
 C4b. $\langle p, q, xp, xq, yp, yq, x(xp + yq + nzs), y(xp + yq + nzs), x^i y^j r \mid 0 \leq i+j \leq n, i, j \geq 0 \rangle, n \geq 0;$
 C4c. $\langle p, q, xp, yp, xq, yq, x(xp + yq + r), y(xp + yq + r), r \rangle;$
 C5a. $\langle p, q, xp - yq, yp, xq, x^i y^j r \mid 0 \leq i+j \leq n, i, j \geq 0 \rangle, n \geq 0;$
 C5b. $\langle p, q, xp - yq, yp, xq, zr, x^i y^j r \mid 0 \leq i+j \leq n, i, j \geq 0 \rangle, n \geq 0;$
 C5c. $\langle p, q + 2xr, xq + x^2r, yp + y^2r, xp - yq, r \rangle;$
 C6a. $\langle p, q, xp - yq, xp + yq + \alpha zs, yp, xq, x^i y^j r \mid 0 \leq i+j \leq n, i, j \geq 0 \rangle, n \geq 0;$
 C6b. $\langle p, q, xp - yq, xp + yq, yp, xq, zr, x^i y^j r \mid 0 \leq i+j \leq n, i, j \geq 0 \rangle, n \geq 0;$
 C6c. $\langle p, q + 2xr, xq + x^2r, yp + y^2r, xp - yp, xp + yq + 2zs, r \rangle;$
 C7a. $\langle p, q, 2xp + \alpha zs, x^2p - xq + \alpha xzs, f(x, y)r \mid f \in V_{\alpha, m_1} + \dots + V_{\alpha, m_k} \rangle, 0 \leq m_1 < \dots < m_k, \text{ where } V_{\alpha, m} = \langle x^i e^{(m-\alpha)y} \mid 0 \leq i \leq m \rangle;$
 C7b. $\langle p, q, xp, x^2p - xq + \alpha xzs, zr, f(x, y)r \mid f \in V_{\alpha, m_1} + \dots + V_{\alpha, m_k} \rangle, 0 \leq m_1 < \dots < m_k;$
 C7c. $\langle p, q, 2xp + r, x^2p - xq + xr, f(x, y)r \mid f \in V_{0, m_1} + \dots + V_{0, m_k} \rangle, 0 < m_1 < \dots < m_k;$
 C7d. $\langle p, q, xp - zr, x^2p - xq - (1+2xz)r, f(x, y)r \mid f \in V_{2, m_1} + \dots + V_{2, m_k} \rangle, 0 \leq m_1 < \dots < m_k;$

- C8a. $\langle p, 2xp + m zr, x^2 p + m x zr, q, x^i f(y)r \mid 0 \leq i \leq m, f \in V_{p(t)} \rangle, m \geq 0, p(t) = t^k + a_{k-1}t^{k-1} + \dots + a_0$, where $V_{p(t)} = \{f \mid f^{(k)} + a_{k-1}f^{(k-1)} + \dots + a_0 f = 0\}$;
- C8b. $\langle p, xp, x^2 p + m x zr, q, zr, x^i f(y)r \mid 0 \leq i \leq m, f \in V_{p(t)} \rangle$;
- C8c. $\langle p, 2xp + r, x^2 p + x r, q, f(y)r \mid f(y) \in V_{p(t)} \rangle$;
- C9a. $\langle p, 2xp + n y q + m zr, x^2 p + n x y q + m x zr, q, x q, \dots, x^n q, x^i y^j r \mid 0 \leq j \leq p, 0 \leq i \leq m - j n \rangle, n \geq 1, m \geq 0, 0 \leq p \leq \lfloor \frac{m}{n} \rfloor$;
- C9b. $\langle p, 2xp + n y q, x^2 p + n x y q + m x zr, q, x q, \dots, x^n q, zr, x^i y^j r \mid 0 \leq j \leq p, 0 \leq i \leq m - j n \rangle, n \geq 1, m \geq 0, 0 \leq p \leq \lfloor \frac{m}{n} \rfloor$;
- C9c. $\langle p, 2xp + n y q, x^2 p + n x y q + x r, q, x q, \dots, x^n q, \frac{\partial}{\partial z} \rangle, n \geq 1$;
- C9d. $\langle p, 2xp + n y q + (kn - 2)zr, x^2 p + n x y q + ((kn - 2)xz + n y^k/k)r, q, x q + y^{k-1}r, x^2 q + 2x y^{k-1}r, \dots, x^n q + n x^{n-1} y^{k-1}r, x^i y^j r \mid 0 \leq i \leq 2n - 2, 0 \leq j \leq k - 2 \rangle, n \geq 1, k \geq 2$;
- C9e. $\langle p, 2xp + n y q + (kn - 2)zr, x^2 p + n x y q + ((kn - 2)xz + n y^k/k)r, q, x q, \dots, x^{n-1} q, x^n q + n x^{n-1} y^{k-1}r, x^i y^j r \mid 0 \leq i \leq n - 2, 0 \leq j \leq k - 1 \rangle, n \geq 2, k \geq 1$;
- C9f. $\langle p, 2xp + y q, x^2 p + x y q + (x + y^2/2)r, q, x q + y r, r \rangle$;
- C9g. $\langle p, xp + y q, x^2 p + 2x y q + (x + y)r, q, x q, x^2 q + x r, r \rangle$;
- C10a. $\langle p, 2xp + (m - n\beta)zr, y q + \beta zr, x^2 p + n x y q + m x zr, q, x q, \dots, x^n q, x^i y^j r \mid 0 \leq j \leq k, 0 \leq i \leq m - n j \rangle, n, m \geq 0, 0 \leq k \leq m/n$;
- C10b. $\langle p, xp, y q, x^2 p + n x y q + m x zr, q, x q, \dots, x^n q, zr, x^i y^j r \mid 0 \leq j \leq k, 0 \leq i \leq m - n j \rangle, n, m \geq 0, 0 \leq k \leq m/n$;
- C10c. $\langle p, xp, y q, x^2 p + n x y q + x r, q, x q, \dots, x^n q, r \rangle, n \geq 0$;
- C10d. $\langle p, xp, y q, x^2 p + x r, q, r, y r, \dots, y^k r \rangle, k \geq 1$;
- C10e. $\langle p, xp, y q + (k + 1)zr + y^{k+1}r, x^2 p, q, r, y r, \dots, y^k \frac{\partial}{\partial z} \rangle, k \geq 0$;
- C10f. $\langle p, xp - y q, y q + k zr, x^2 p + n x y q + ((kn - 2)xz + n y^k/k)r, q, x q + y^{k-1}r, x^2 q + 2x y^{k-1}r, \dots, x^n q + n x^{n-1} y^{k-1}r, x^i y^j r \mid 0 \leq i \leq 2n - 2, 0 \leq j \leq k - 2, n \geq 1, k \geq 2$;
- C10g. $\langle p, xp - y q, y q + k zr, x^2 p + n x y q + ((kn - 2)xz + n y^k/k)r, q, x q, \dots, x^{n-1} q, x^n q + n x^{n-1} y^{k-1}r, x^i y^j r \mid 0 \leq i \leq n - 2, 0 \leq j \leq k - 1, n \geq 2, k \geq 1$.

A.7. Additional algebras in real case.

- P4'. $\langle p, q, r, xp - zr, xq + yr, yp + zq, xp + yq + zr, 2zU + Sp, 2yU - Sq, 2xU + Sr \rangle, U = xp + yq + zr, S = y^2 - 2xz$;
- P5'. $\langle p, q, r, xp - zr, xq + yr, yp + zq, xp + yq + zr \rangle$;
- P6'. $\langle xq - yp, xr - zp, yr - zq, Sp - 2xU, Sq - 2yU, Sr - 2zU \rangle$, where $S = 1 + x^2 + y^2 + z^2, U = xp + yq + zr$;
- P6''. $\langle xp - zr, xq + yr, yp + zq, 2zU + Sp, 2yU - Sq, 2xU + Sr \rangle, U = xp + yq + zr, S = y^2 - 2xz + 1$;
- P6'''. $\langle xp - zr, xq + yr, yp + zq, 2zU + Sp, 2yU - Sq, 2xU + Sr \rangle, U = xp + yq + zr, S = y^2 - 2xz - 1$;
- P7'. $\langle p, q, r, xp - zr, xq + yr, yp + zq \rangle$;
- B1'. $\langle xp - yq + r, (1 + x^2 - y^2)p + 2x y q + 2y r, 2x y p + (1 - x^2 + y^2)q - 2x r \rangle$;
- B2'. $\langle p, q, xp + yq + \cos(\alpha)r, xq - yp + \sin(\alpha)r, (x^2 - y^2)p + 2x y q + 2(\cos(\alpha)x + \sin(\alpha)y)r, -2x y p + (x^2 - y^2)q + 2(\sin(\alpha)x - \cos(\alpha)y)r \rangle, \alpha \sim -\alpha$;
- B4'. (primitive over \mathbb{R})
- C2a'. $\langle xq - yp, (\pm 1 + x^2 - y^2)p + 2x y q, 2x y p + (\pm 1 - x^2 + y^2)q, f(x, y)r \mid f \in V_{n_1} + \dots + V_{n_k} \rangle, 0 \leq n_1 < \dots < n_k$. Here V_n is an $(2n + 1)$ dimensional

vector space generated by $P_n(\frac{x^2+y^2\pm 1}{x^2+y^2\pm 1})$, where $P_n(z)$ is the n -th Legendre polynomial:

$$P_n(z) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dz^n} (1 - z^2)^n.$$

C2b'. $\langle xq - yp, (\pm 1 + x^2 - y^2)p + 2xyq, 2xyp + (\pm 1 - x^2 + y^2)q, zrf(x, y)r \mid f \in V_{n_1} + \dots + V_{n_k}, 0 \leq n_1 < \dots < n_k \rangle$;

C2c'. $\langle xq - yp + r, (\pm 1 + x^2 - y^2)p + 2xyq + 2yr, 2xyp + (\pm 1 - x^2 + y^2)q - 2xr, f(x, y)r \mid f \in V_{n_1} \oplus \dots \oplus V_{n_k}, 0 \leq n_1 < \dots < n_k \rangle$;

C3a'. $\langle p, q, xp + yq + n_zr, xq - yp, (x^2 - y^2)p + 2xyq + 2nxzr, -2xyp + (x^2 - y^2)q - 2nyzr, x^i y^j (x^2 + y^2)^k \mid 0 \leq i + j + k \leq n \rangle$;

C3b'. $\langle p, q, xp + yq, xq - yp, (x^2 - y^2)p + 2xyq + 2nxzr, -2xyp + (x^2 - y^2)q - 2nyzr, zr, x^i y^j (x^2 + y^2)^k \mid 0 \leq i + j + k \leq n \rangle$;

C3c'. $\langle p, q, r, xp + yq, xq - yp, (x^2 - y^2)p + 2xyq + 2(\cos(\alpha)x + \sin(\alpha)y)r, -2xyp + (x^2 - y^2)q + 2(\sin(\alpha)x - \cos(\alpha)y)r \rangle, \alpha \sim -\alpha$.

A.8. Lie algebras with one-dimensional invariant foliations of type D.

Lie algebras D1-D18 are obtained from the Lie algebras of vector fields on the plane by adding the subalgebra $\langle r, zr, z^2r \rangle$.

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